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Positivity of characteristic forms *via* pointwise universal push-forward formulæ

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Abstract

Given a Hermitian holomorphic vector bundle over a complex manifold, consider its flag bundles with the associated universal vector bundles endowed with the induced metrics. We show that the universal formula for the push-forward of a homogeneous polynomial in the Chern classes of the universal vector bundles also holds pointwise at the level of Chern forms in this Hermitianized situation.

As an application, we obtain the positivity of several polynomials in the Chern forms of Griffiths semipositive vector bundles not previously known. This gives new evidences towards a conjecture proposed by Griffiths, which has raised interest in the past as well as in recent years. This conjecture can be interpreted as a pointwise Hermitianized version of the Fulton–Lazarsfeld theorem on numerically positive polynomials for ample vector bundles.

Contents

Introduction

The main purpose of this thesis is the study of the characteristic differential forms associated to Hermitian holomorphic vector bundles over complex manifolds. More precisely, this thesis addresses two issues. The first one deals with the extension to the level of differential forms of certain push-forward formulæ for flag bundles valid in cohomology. The second one concerns the positivity of the characteristic differential forms arising from the Chern curvature of positive vector bundles.

The connection between these two issues consists in the use of the pointwise push-forward formulæ obtained to deduce the positivity of several characteristic forms associated to positive vector bundles.

Preliminaries

Let X be a complex manifold of dimension n and let $E \to X$ be a holomorphic vector bundle of rank $r \geq 2$. Once fixed a sequence ρ of dimensions

$$0 = \rho_0 < \rho_1 < \cdots < \rho_{m-1} < \rho_m = r,$$

one can consider the (incomplete if $m < r$, or complete if $m = r$) flag bundle $\pi_\rho \colon \mathbb{F}_\rho(E) \to X$, which is naturally endowed with a filtration

$$(0) = U_{\rho,0} \subset U_{\rho,1} \subset \cdots \subset U_{\rho,m-1} \subset U_{\rho,m} = \pi_\rho^* E$$

of $m+1$ tautological vector bundles where, for $0 \leq l \leq m$, the rank of $U_{\rho,l} \to \mathbb{F}_\rho(E)$ is ρ_l. Out of this filtration, one can form the universal quotient bundles

$$U_{\rho,l} \big/ U_{\rho,\ell} \to \mathbb{F}_\rho(E), \quad 0 < \ell < l \leq m.$$

Both tautological bundles and universal quotients are called *universal vector bundles* over $\mathbb{F}_\rho(E)$. Let $\mathcal{E}_1, \dots, \mathcal{E}_N$ be an enumeration of all the universal vector bundles, where $N = \binom{m+1}{2}$. We denote by $r_1, \dots, r_N$ the corresponding ranks. Finally, for $1 \leq j \leq N$, consider the corresponding Chern classes $c_1(\mathcal{E}_j), \dots, c_{r_j}(\mathcal{E}_j)$ in the cohomology of $\mathbb{F}_\rho(E)$.

Now, given a homogeneous polynomial F in $r_1 + \cdots + r_N$ variables of degree $d_\rho + k$, where d_ρ is the relative dimension of the proper holomorphic submersion π_ρ and $0 \leq k \leq n$, the proper push-forward

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1), \dots, c_\bullet(\mathcal{E}_N))$$

gives a cohomology class in $H^{2k}(X)$ which of course needs to be a characteristic class for E. It is then a natural issue (which has been considered and settled by several authors in different degree of generality) to try to determine a closed formula to express this class more or less explicitly as a polynomial, call it Φ, in the Chern (or Segre) classes of E.

There is a vast literature concerning push-forward formulæ (which are also called *Gysin formulæ*) for flag bundles, and the approaches followed are several. For instance, Gysin formulæ for flag bundles are given by:

- [Qui69, Dam73, AC87] by using Grothendieck residues;
- [BS12, Tu17, Zie18] by using residues at infinity;
- [Bri96, PR97] by using symmetrizing operators;
- [JLP81] by using Schur functions.

See also the books [FP98, Man98].

In this thesis, we shall consider for our purposes the incarnation of such a formula given by Darondeau and Pragacz in [DP17]. For similar formulæ see [Ilo78]. We also shall use a Gysin formula for Grassmann bundles given in [KT15].

Pointwise push-forward formulæ for flag bundles

Alongside the cohomological situation described above, one can ask the analogue in the Hermitian setting as follows. Suppose that E is moreover endowed with a smooth Hermitian metric h. From the Chern curvature tensor $\Theta(E,h)$, let us consider the corresponding Chern forms on X defined, for $0 \leq s \leq r$, as

$$c_s(E,h) = \operatorname{tr}_{\operatorname{End}(\Lambda^s E)}\left(\bigwedge^s \frac{i}{2\pi}\Theta(E,h)\right).$$

By the Chern–Weil theory the form $c_s(E,h)$ represents the Chern class $c_s(E)$ of the vector bundle E.

The universal vector bundles $\mathcal{E}_j$'s considered before inherit, being sub-bundles of $\pi_\rho^* E$ or quotients of them, natural Hermitian metrics H_j's. Thus, the classes $c_1(\mathcal{E}_j), \dots, c_{r_j}(\mathcal{E}_j)$ now have special representatives $c_1(\mathcal{E}_j, H_j), \dots, c_{r_j}(\mathcal{E}_j, H_j)$ given by the Chern forms of their induced Hermitian metrics.

Given the homogeneous polynomial F above, one can formally compute it using the $c_\bullet(\mathcal{E}_j, H_j)$'s as variables to get a closed $(d_\rho + k, d_\rho + k)$-form on $\mathbb{F}_\rho(E)$, which can be pushed-forward on X *via* integration along the fibers to obtain a (k,k)-form

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \dots, c_\bullet(\mathcal{E}_N, H_N))$$

on X. Such a form is of course a special representative for the class

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1), \dots, c_\bullet(\mathcal{E}_N)) = \Phi(c_\bullet(E)),$$

where Φ is the polynomial given by universal Gysin formulæ mentioned above.

Certainly, one can also formally evaluate Φ in the Chern forms of (E,h), thus obtaining another special representative, namely $\Phi(c_\bullet(E,h))$, for the cohomology class $\Phi(c_\bullet(E))$. Therefore, *a priori*, the forms $(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \dots, c_\bullet(\mathcal{E}_N, H_N))$ and $\Phi(c_\bullet(E,h))$ differ by an error term which is an exact $2k$-form.

The first central result of this thesis (see Theorem 2.16) can be now summarized in the following.

Theorem A. *We have the equality of differential forms*

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \dots, c_\bullet(\mathcal{E}_N, H_N)) = \Phi(c_\bullet(E, h)).$$

So, in fact, there is no error term at all. In other words, the universal Gysin formulæ to compute the push-forwards in cohomology from the flag bundle can be now used *verbatim* to compute pointwise the push-forwards for differential forms constructed from the Chern–Weil theory in the Hermitian situation.

A remarkable special case of Theorem A, for which we dedicate a separate description due to its applications in positivity as we explain below, occurs when we consider the m line bundles

$$Q_{\rho,l} := \det\left(U_{\rho,m-l+1} \big/ U_{\rho,m-l} \right) \to \mathbb{F}_\rho(E), \quad 1 \le l \le m,$$

which are the determinants of the successive quotients of the tautological bundles.

The Chern classes $c_1(Q_{\rho,1}), \dots, c_1(Q_{\rho,m})$ have special representatives denoted by $\Xi_{\rho,1}, \dots, \Xi_{\rho,m}$, which are the Chern curvatures of the determinants of the natural Hermitian metrics induced by h mentioned before. Equivalently, these representatives are the first Chern forms of the Hermitian metrics induced on the successive quotients.

As before, by taking a homogeneous polynomial G in m variables of degree $d_\rho + k$, where $0 \le k \le n$, the universal Gysin formulæ for flag bundles provide us a polynomial Γ in the Chern classes of E such that the equality

$$(\pi_\rho)_* G(c_1(Q_{\rho,1}), \dots, c_1(Q_{\rho,m})) = \Gamma(c_\bullet(E))$$

between cohomology classes holds.

Moving to the representatives level, we prove the following result (see Theorem 2.9).

Theorem B. *We have the equality of differential forms*

$$(\pi_\rho)_* G(\Xi_{\rho,1}, \dots, \Xi_{\rho,m}) = \Gamma(c_\bullet(E, h)).$$

In the special case of the projectivized bundle $\mathbb{P}(E)$ of lines (resp. $\mathbb{P}(E^\vee)$ of hyperplanes) in E, which, in our notation, corresponds to the sequence ρ of dimensions $(0, 1, r)$ (resp. $(0, r-1, r)$), Theorem B generalizes previous results of [Mou04, Gul12, Div16].

More precisely, [Mou04, Proposition 6] proves, by explicit calculations, the push-forward identity in the case $G(\Xi_{\rho,\bullet})$ a power of the first Chern form of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ (resp. $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$) equipped with the induced metric. In such case, the push-forward gives the (signed) Segre forms of (E, h), which are special representatives for the Segre classes of E. Alternative proofs of the push-forward formula for projectivized bundles are given in [Gul12, Proposition 3.1] (whose approach is followed in this thesis) and in [Div16, Proposition 1.1] (by pointwise computations).

Theorem B also generalizes the recent result [Fin21, Theorem 3.18] which is a special case of our theorem but for complete flag bundles and where, in our notations, G is taken to be just a monomial with some specific decreasing degrees.

Although, as already mentioned, Theorem B is a particular case of Theorem A, we have chosen here to state them as two separate results by several reasons. Beside the fact that, chronologically, Theorem B came before Theorem A, the proof of the former is obtained by means of explicit computations, while the proof of the latter does not use explicit formulæ. This because, in the end, we do not necessarily need to perform explicitly integration along the fibers to obtain the push-forward formula. The second reason is that the proof of Theorem B is based on the curvature formulæ for universal line bundles over $\mathbb{F}_\rho(E)$ provided by Demailly in [Dem88a, Formula (4.9)]. However, another main result of this thesis is the calculation of the Chern curvature (at a point) of any universal vector bundle over $\mathbb{F}_\rho(E)$, and the explicit expression of these curvatures (which we provide in Theorem 2.11) are needed to prove Theorem A. The final reason is that we use some of the Gysin formulæ of Theorem B for applications in positivity theory as the following section explains.

Applications: positivity of characteristic forms

The last part of the thesis is devoted to an application of Theorems A and B to a positivity issue, in the same spirit of [Gul12], as follows.

Denote by T_X the holomorphic tangent bundle of X, and suppose that $(E,h) \to X$ is a *Griffiths semipositive* (resp. *Griffiths positive*) vector bundle. This means that for every $x \in X$, $v \in E_x$, $\tau \in T_{X,x}$ the Chern curvature tensor of (E,h) satisfy

$$\langle \Theta(E,h)_x \cdot v, v\rangle_h(\tau,\bar\tau) \geq 0$$

(resp. > 0, and $= 0$ if and only if v or τ is the zero vector).

Recall also that a holomorphic vector bundle $\mathcal{V} \to X$ of rank r is called *ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1) \to \mathbb{P}(\mathcal{V}^\vee)$ is ample.

It is well known that a Griffiths positive vector bundle over a compact complex manifold is ample (the converse is not known in general, but it is a conjecture) and that a globally generated vector bundle (*i.e*, a bundle whose fibers are spanned by the global holomorphic sections) can be equipped with a Hermitian metric which makes it Griffiths semipositive.

It is natural to expect that certain conditions of positivity on the vector bundle (such as ampleness or Griffiths positivity) impose, in turn, the positivity of objects that derive from it.

In the seminal paper [Gri69] it is raised the problematic of determine which characteristic forms built from the Chern curvature of (E,h) are positive, and some partial result is given. More precisely, it is asked (and expected) whether the *Schur forms* and hence, their positive linear combinations, are positive differential forms. See Section 1.4.1 for definitions of the three main notions of positivity for differential forms. See also [HK74, Dem12].

We now give a more detailed description of the Griffiths' question, which is known also as the "Griffiths' conjecture on the positivity of Chern–Weil forms".

Fix $k \in \mathbb{N}$ and denote by $\Lambda(k,r)$ the set of partitions of k in r parts, *i.e.*, the set of all $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{N}^k$ such that

$$r \geq \sigma_1 \geq \cdots \geq \sigma_k \geq 0, \quad |\sigma| = \sum_{j=1}^{k} \sigma_j = k.$$

To $\sigma \in \Lambda(k, r)$ we associate the *Schur polynomial* $S_\sigma \in \mathbb{Z}[c_1, \ldots, c_r]$ of weighted degree $2k$ (we regard c_s as having degree $2s$), which is defined as

$$S_\sigma(c_1, \ldots, c_r) := \det \left(c_{\sigma_i + j - i}\right)_{1 \leq i, j \leq r}.$$

By convention, $c_0 = 1$ and $c_s = 0$ if $s < 0$ or $s > r$.

As $\sigma \in \Lambda(k, r)$ varies, the Schur polynomials form a basis for the $\mathbb{Q}$-vector space of degree $2k$ weighted homogeneous polynomials in r variables, and the product of two Schur polynomials is a positive linear combination of Schur polynomials. Hence, Schur polynomials generate a positive convex cone closed under product, which is called *Schur cone*. Following [Gri69] we call it $\Pi(r)$ (however, remark that this is not exactly the positive cone considered by Griffiths, but they coincides *a posteriori* thanks to the work of [FL83]).

Taking the vector bundle $(E, h) \to X$ as before, we can formally evaluate the Schur polynomial S_σ on the Chern forms of (E, h), obtaining a characteristic differential form on X. This is the so called *Schur form* of (E, h) associated to σ; we denote it by $S_\sigma(E, h)$. Clearly, $S_\sigma(E, h)$ represents the *Schur class* $S_\sigma(E)$, which is the cohomology class of X obtained by formally evaluating the polynomial S_σ in the Chern classes of E.

Significant examples of Schur classes (resp. forms) are:

- the Chern classes (resp. forms), corresponding to $\sigma = (k, 0, \ldots, 0)$;
- classes (resp. forms) of type $c_1 c_{k-1} - c_k$, corresponding to $\sigma = (k-1, 1, 0, \ldots, 0)$;
- the signed Segre classes (resp. forms), corresponding to $\sigma = (1, \ldots, 1)$.

To fix the terminology, we call the *Schur cone* of E (resp. of (E, h)) the positive convex cone formed by all the polynomials in $\Pi(r)$ formally evaluated in the Chern classes (resp. forms) of E (resp. of (E, h)). We denote it by $\Pi(E)$ (resp. $\Pi(E, h)$).

Griffiths' conjecture (which we now state with a modification with respect to the original statement in [Gri69]) can be now summarized in the following.

Conjecture ([Gri69])**.** Given any Griffiths semipositive Hermitian holomorphic vector bundle $(E, h) \to X$ of rank r and a polynomial $P \in \Pi(r)$, then the differential form $P(c_\bullet(E, h)) \in \Pi(E, h)$ is positive (in the weak sense of differential forms).

In other words, the conjecture asserts that the cone $\Pi(E, h)$ consists of positive differential forms and that, in particular, the Schur forms $S_\sigma(E, h)$'s are positive.

Griffiths' conjecture was thus a differential, pointwise forerunner of the Fulton–Lazarsfeld theorem [FL83], which is a cohomological global statement, that characterizes precisely all numerically positive polynomials for ample vector bundles. The Fulton–Lazarsfeld theorem indeed characterizes them exactly as the positive linear combinations of Schur polynomials, thus extending the previous works

[Kle69, BG71, Gie71, UT77]. See [DPS94, Theorem 2.5] for an extension of [FL83, Theorem 3.1] to nef vector bundles on compact Kähler manifolds.

Observe that a cohomology class which can be represented by a positive differential form is numerically positive. Thus, an answer in the affirmative to Griffiths' conjecture would give a stronger Fulton–Lazarsfeld-type statement under the stronger (but conjecturally equivalent) hypothesis of positivity in the sense of Griffiths.

Up to now, very little is known about this question beside the trivial case of (any power of) the first Chern form. Griffiths proved in [Gri69, Appendix to §5.(b)] that the second Chern form of a rank 2 Griffiths positive vector bundle is positive. However, see [Gri69, p. 247], Griffiths himself deemed difficult to adapt his proof to the general case.

In the last few years there has been a renewed interest around Griffiths' question, as important partial results have appeared in the literature. Guler proved in [Gul12, Theorem 1.1] the positivity of the already mentioned signed Segre forms.

Strengthening the hypotheses with stronger notions of positivity, such as (dual) Nakano semipositivity (see for instance [Dem12] for the definitions), the conjecture was proven in [Li21, Proposition 3.1] and in [Fin21, Theorem 1.1]. It is also worth mentioning that Griffiths himself proved his own conjecture in [Gri69, Proof of Theorem D] for globally generated vector bundles. However, the problem still remains open in the general case. See also, for instance, [Div16, Pin18, RT21, Xia22] for further related results.

In Section 3.4.1 we provide an overview on the state of the art of Griffiths' conjecture, focusing, in particular, on the recent progress. Primarily, the purpose of such overview is to clarify which types of positivity (among those in Definition 1.15) have been obtained for the Schur forms of positive vector bundles. Indeed, the terminology used in the literature is currently not always standard, and sometimes may lead to some confusion.

Here, as a consequence of our Theorem B, we are able to establish the positivity of several new positive linear combinations of Schur forms, cf. Section 3.2. Namely, thanks to the curvature formulæ given in [Dem88a, Formula (4.9)] coupled with our universal pointwise push-forward formulæ, we obtain the following result (see Theorem 3.6).

Theorem C. *In the notation as above, let $(E,h) \to X$ be Griffiths semipositive. Given a weight $\mathbf{a} \in \mathbb{Z}^m$ such that $a_1 \geq \cdots \geq a_m \geq 0$, we have that the characteristic forms*

$$(\pi_\rho)_*(a_1\, \Xi_{\rho,1} + \cdots + a_m\, \Xi_{\rho,m})^{\wedge(d_\rho+k)}$$

are strongly positive (k,k)-forms on X, and moreover positive linear combinations of Schur forms of (E,h).

This confirms Griffiths' conjecture (in even a stronger way) for those differential forms belonging to the sub-cone, which we denote by $\mathcal{F}(E,h)$, of $\Pi(E,h)$ spanned by all the possible wedge products of all the possible push-forwards of type as in Theorem C (see Section 3.2.1 for several concrete examples).

The sub-cone $\mathcal{F}(E,h)$ contains in particular the signed Segre forms, therefore Theorem C generalizes [Gul12, Theorem 1.1].

Observe also that it is in some sense more natural to obtain that these forms are strongly positive rather than merely positive. This is because, as already said, polynomials in $\Pi(r)$ are stable under products and so do strongly positive forms, while a product of positive forms is not necessarily still positive.

As a byproduct of the proof of Theorem C, we also obtain a partial answer to a conjecture raised in [Xia22, Conjecture 1.4]. It asks whether every Schur class $S_\sigma(\mathcal{V})$ of an ample vector bundle $\mathcal{V}$ over a projective variety X does admit a positive representative, and [Xia22, Theorem A] gives an affirmative answer if $|\sigma| = \dim X - 1$. Our result (see Theorem 3.8) shows that for each $\mathbf{a} \in \mathbb{Z}^m$ that satisfy the condition $a_1 \geq \cdots \geq a_m \geq 0$, the cohomology classes

$$(\pi_\rho)_*(a_1\, c_1(Q_{\rho,1}) + \cdots + a_m\, c_1(Q_{\rho,m}))^{d_\rho + k}$$

belong the Schur cone $\Pi(E)$ and admit a strongly positive representative.

Afterwards, we focus on some differential forms which in general do not belong to the sub-cone $\mathcal{F}(E,h)$.

First, we show how to obtain the positivity of the second Chern form $c_2(E,h)$ in any rank, if (E,h) is Griffiths positive. This can be easily obtained by adapting the above mentioned Griffiths' result on the positivity of $c_2(E,h)$ in rank 2. At the best of our knowledge, this was first observed by Guler in [Gul06], but just as a statement without proof. Here we give full details and, furthermore, we extend such result to the semipositive context in Theorem 3.11.

Subsequently, we focus on the Griffiths' conjecture in rank 3, for which we prove the following result (see Theorem 3.12).

Theorem D. *In the notation as above, let $(E,h) \to X$ be Griffiths semipositive of rank 3. Then the Schur form*

$$S_{(2,1,0)}(E,h) = c_1(E,h) \wedge c_2(E,h) - c_3(E,h)$$

is a positive $(3,3)$-forms on X.

The starting point to obtain this result is to consider a variant of Theorem A (which we establish in Formula (2.19)) in order to get the form $S_{(2,1,0)}(E,h)$ as a push-forward from the complete flag bundle $\mathbb{F}(E)$; cf. with Proposition 2.19 where we prove that each Schur form can be obtained as a push-forward. But, at the best of our knowledge, we are not able to deduce the positivity of this form instantly from that push-forward. Hence, our idea is to factorize through the projectivized bundle $\mathbb{P}(E)$. In this way, we find that $S_{(2,1,0)}(E,h)$ can be obtained also by pushing forward a positive form on $\mathbb{P}(E)$, leading us to prove Theorem D.

As a consequence of Theorem D, we find that the following chain of inequalities

$$c_1(E,h)^{\wedge 3} \geq c_1(E,h) \wedge c_2(E,h) \geq c_3(E,h)$$

holds for Griffiths semipositive vector bundles of rank 3 over complex manifolds (see Corollary 3.16). Similar inequalities have appeared, for instance, in [Li21, Theorem 3.2] if h is dual Nakano semipositive, and they give a metric counterpart of some well-known inequalities between Chern numbers valid for nef vector bundles on compact Kähler manifolds (see, [FL83, DPS94, LZ20] and [Li21, Remark 3.3]). In the setting of Theorem D, if X is compact (not necessarily Kähler) of dimension 3, it follows from the positivity of $S_{(2,1,0)}(E, h)$ that the Chern numbers of E satisfy the relation

$$c_1{}^3 \geq c_1 c_2 \geq c_3.$$

Certainly, it is clear that any differential form in the cone $\mathcal{F}(E, h)$, for which we proved the strong positivity in Theorem C, gives certain inequalities between Chern forms/numbers. However, we have preferred here to highlight the chain of inequalities coming from $S_{(2,1,0)}(E, h)$ since it was previously known, with different assumptions on E and X, in works such as [DPS94, LZ20].

In the last part of the thesis (see Section 3.4) we collect some concluding remarks and open questions which may serve for further developments of this topic.

Concluding note

Some of the main results of this thesis are next to publication. More precisely, Theorem B and Theorem C are respectively the main theorem and main application of the author's paper [DF22] joint with S. Diverio. Theorem D is the main result of the paper [Fag22] by the author. Finally, the curvature computations of the universal vector bundles in Chapter 2 and Theorem A are results obtained by the author shortly before the redaction of this thesis. They are currently in preparation in view of a future publication.

Chapter 1

Preliminary notions

In this chapter we introduce the concepts necessary to state the main results of this thesis. Namely, in Section 1.2 we recall the notions of flag bundle ([Dem88a, Dem88b, Ful98]) and universal vector bundles, which are the main objects of our results in Chapter 2. Then, in Section 1.3 we give the definition of Schur polynomial ([FL83, DPS94, Ful97]) and some notions related to Schur functions ([DP17]). Finally, in Section 1.4 we recall some of the positivity notions for differential forms ([HK74, Dem12]) and vector bundles ([Gri69, Laz04, Dem12, Fin21]) needed for our results in Chapter 3. Several further references are cited during the exposition, in particular for the proofs of the results that we recall.

1.1 Introductory notation

Let X be a complex manifold of dimension n.

If $E \to X$ is a complex vector bundle and $0 \leq p, q \leq n$, then $\mathcal{A}^{p,q}(X, E)$ stands for the space $C^\infty\big(X, \Lambda^{p,q} T_X^\vee \otimes E\big)$ of differential (p,q)-forms on X with values in E. In particular, $\mathcal{A}^{p,q}(X)$ denotes the space $C^\infty\big(X, \Lambda^{p,q} T_X^\vee\big)$ of differential (p,q)-forms on X. Similarly, $\mathcal{A}^k(X)$ stands for the space of differential k-forms on X.

In this section we suppose that the vector bundle $E \to X$ is holomorphic of rank r and equipped with a Hermitian metric h. We denote by

$$\Theta(E, h) \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$$

the Chern curvature tensor of the Hermitian vector bundle (E, h), where

$$\mathrm{End}(E) \cong E^\vee \otimes E$$

is the endomorphisms bundle of E and $E^\vee$ is the dual bundle of E. Here we prefer to use the symbol $\cdot^\vee$ for the dual in order to avoid confusion with the symbol $\cdot^*$ of pull-back.

Recall that, out of the Chern curvature $\Theta(E, h)$ of the Hermitian vector bundle (E, h), one can construct the *Chern forms* of (E, h) on X. They are defined, for $0 \leq s \leq r$, as

$$c_s(E, h) = \mathrm{tr}_{\mathrm{End}(\Lambda^s E)} \left(\bigwedge^s \frac{i}{2\pi} \Theta(E, h) \right).$$

By the Chern–Weil theory the form $c_s(E,h) \in \mathcal{A}^{s,s}(X)$ is d-closed, real and represents the Chern class $c_s(E) \in H^{2s}(X,\mathbb{Z})$ of the vector bundle E.

Fix a point $x_0 \in X$ and local holomorphic coordinates $z = (z_1,\dots,z_n)$ on an open set of X centered at x_0. Recall that a local holomorphic frame $(e_1,\dots,e_r)$ of E centered at x_0 is called a *normal coordinate frame* at x_0 (see, [Dem12, (12.10) Proposition]) if

$$\langle e_\alpha(z), e_\beta(z)\rangle_h = \delta_{\alpha\beta} - \sum_{1\le j,k\le n} c_{jk\alpha\beta}\, z_j \bar z_k + \mathrm{O}(|z|^3),$$

where the $c_{jk\alpha\beta}$'s are the coefficients of the Chern curvature tensor

$$\Theta(E,h)_{x_0} = \sum_{1\le\alpha,\beta\le r}\ \sum_{1\le j,k\le n} c_{jk\alpha\beta}\, dz_j \wedge d\bar z_k \otimes e_\alpha^\vee \otimes e_\beta$$

expressed with respect to the frame $(e_1,\dots,e_r)$.

Since $i\Theta(E,h)$ is a $(1,1)$-form with values in the bundle $\mathrm{Herm}(E,h)$ of Hermitian endomorphisms of E, we have the symmetry relation $\bar c_{jk\alpha\beta} = c_{kj\beta\alpha}$.

In this notation, the (β,α)-entry of the matrix associated to $\Theta(E,h)_{x_0}$ with respect to $(e_1,\dots,e_r)$ in the coordinate open set considered is the $(1,1)$-form

$$\Theta_{\beta\alpha} := \sum_{1\le j,k\le n} c_{jk\alpha\beta}\, dz_j \wedge d\bar z_k.$$

Push-forward given by integration along fibers

Let $\pi\colon S \to X$ be a proper holomorphic submersion of complex manifolds with relative dimension s. Then, for $k = 0,\dots,\dim_{\mathbb{R}} X$ the map

$$\pi_*\colon \mathcal{A}^{2s+k}(S) \to \mathcal{A}^k(X)$$

is the *push-forward* through π, which is given by integration along the fibers of the submersion. Given a differential form η on S, the push-forward can be computed at $x \in X$ by locally splitting the variables of the fiber $\pi^{-1}(x)$ as (x,y) in such a way that

$$\pi_*\eta(x) = \int_{y\in\pi^{-1}(x)} \eta(x,y),$$

where the right hand side stands for the integral of $\eta(x,y)$ performed only on those differentials related to the variable y. For precise definitions, properties and results about push-forwards of differential forms we refer, mainly, to [BT82, Dem12]. We just recall here the so-called *projection formula*, which relates push-forwards and pull-backs. For each differential form α on S and β on X, the following equality

$$\pi_*\left(\alpha \wedge \pi^*\beta\right) = \pi_*\alpha \wedge \beta$$

holds.

We also use the same notation for the map

$$\pi_*\colon H^{2s+k}(S) \to H^k(X)$$

induced in cohomology by integration along fibers of π. These maps are also called *Gysin* homomorphisms (see, for instance, [Ful98]).

1.2 Flag bundles

This section mainly follows [Dem88a, Dem88b]. See also, for instance, [Ful97, Ful98] for further references on flag manifolds and bundles.

Let X be a complex manifold of dimension n and let $E \to X$ be a holomorphic vector bundle of rank r. Given a natural number $m > 0$, fix a sequence of integers $\rho = (\rho_0, \dots, \rho_m)$ of the form

$$0 = \rho_0 < \rho_1 < \cdots < \rho_l < \cdots < \rho_{m-1} < \rho_m = r. \qquad (\star)$$

The *flag bundle of type ρ associated to E* is the holomorphic fiber bundle

$$\pi_\rho \colon \mathbb{F}_\rho(E) \to X$$

where the fiber over $x \in X$ is the flag manifold $\mathbb{F}_\rho(E_x)$, whose points $\mathbf{f}_{x,\rho}$ are the flags of the form

$$\{0_x\} = V_{x,\rho_0} \subset V_{x,\rho_1} \subset \cdots \subset V_{x,\rho_l} \subset \cdots \subset V_{x,\rho_{m-1}} \subset V_{x,\rho_m} = E_x$$

where, for each $0 \leq l \leq m$, $\dim_{\mathbb{C}} V_{x,\rho_l} = \rho_l$.

Over $\mathbb{F}_\rho(E)$ we have a tautological filtration

$$(0) = U_{\rho,0} \subset U_{\rho,1} \subset \cdots \subset U_{\rho,l} \subset \cdots \subset U_{\rho,m-1} \subset U_{\rho,m} = \pi_\rho^* E \qquad (1.1)$$

of vector sub-bundles of $\pi_\rho^* E$, where, for every $0 \leq l \leq m$, the fiber of $U_{\rho,l}$ over the point $(x, \mathbf{f}_{x,\rho}) \in \mathbb{F}_\rho(E)$ is V_{x,ρ_l}. Therefore, the vector bundle $U_{\rho,l} \to \mathbb{F}_\rho(E)$ has rank ρ_l. Finally, denote by d_ρ the relative dimension of the proper holomorphic submersion π_ρ.

Example 1.1. If ρ is the complete sequence $(0, 1, \dots, r-1, r)$, then $\mathbb{F}_\rho(E)$ is the *complete* flag bundle associated to E. Observe that this occurs when $m = r$. In this case, we shall drop the subscript ρ simply writing

$$\pi \colon \mathbb{F}(E) \to X$$

and denoting by d the relative dimension of π. Accordingly, for $0 \leq l \leq r$, the tautological filtration (1.1) is written as

$$(0) = U_0 \subset U_1 \subset \cdots \subset U_l \subset \cdots \subset U_{m-1} \subset U_r = \pi^* E.$$

Now, fix $x \in X$ and set $V := E_x$. By using the description of the flag manifold $\mathbb{F}_\rho(V)$ as a homogeneous spaces (see, for instance, [Dem88a, Dem88b, Ful97]), we compute the relative dimension d_ρ of the flag bundle $\mathbb{F}_\rho(E)$ in the following.

Proposition 1.2. *For $\rho = (\rho_0, \dots, \rho_m)$ satisfying Condition $\star$, it holds that*

$$d_\rho = \frac{r^2 - \sum_{j=1}^m (\rho_j - \rho_{j-1})^2}{2}. \qquad (1.2)$$

In particular, if ρ is the complete sequence, it holds that

$$d = \frac{r(r-1)}{2}. \qquad (1.3)$$

Proof. Let $\mathbf{f}_\rho \in \mathbb{F}_\rho(V)$ be the flag identified by a basis $(\varphi_1, \dots, \varphi_r)$ of V as

$$\{0_V\} \subset V_{\rho_1} \subset \cdots \subset V_{\rho_l} = \langle \varphi_{r-\rho_l+1}, \varphi_{r-\rho_l+2}, \dots, \varphi_r \rangle \subset \cdots \subset V_{\rho_{m-1}} \subset V.$$

In order to emphasize the basis $(\varphi_1, \dots, \varphi_r)$ defining the flag $\mathbf{f}_\rho$, it is common to use the notation $[\varphi_1, \dots, \varphi_r]_\rho$.

The general linear group $\mathrm{GL}(r, \mathbb{C})$ acts on the right of $\mathbb{F}_\rho(V)$ as follows:

$$([\varphi_1, \dots, \varphi_r]_\rho, A) \mapsto \left[\sum_{\lambda=1}^{r} a_{\lambda 1} \varphi_\lambda, \dots, \sum_{\lambda=1}^{r} a_{\lambda r} \varphi_\lambda \right]_\rho$$

where the $a_{\lambda\mu}$'s are the entries of the matrix in A acting on the flag $\mathbf{f}_\rho$.

The stabilizer of $\mathbf{f}_\rho$ with respect to this action is the parabolic subgroup P_ρ of matrices $(a_{\lambda\mu})$ with $a_{\lambda\mu} = 0$ for all λ, μ such that there is an integer $\ell = 1, \dots, m-1$ with $\lambda \leq r - \rho_{m-\ell} < \mu$. Note that if ρ is the complete sequence then P_ρ is the Borel subgroup of lower triangular matrices.

Therefore, there is a diffeomorphism

$$\mathbb{F}_\rho(V) \cong \mathrm{GL}(r, \mathbb{C})/P_\rho$$

and it holds that

$$\dim \mathbb{F}_\rho(V) = \dim \mathrm{GL}(r, \mathbb{C}) - \dim P_\rho.$$

For a matrix $(a_{\lambda\mu}) \in P_\rho$, the above condition for which $a_{\lambda\mu} = 0$ gives, for each $j = 1, \dots, m-1$ and $s = j, \dots, m-1$, an upper block of zeros in $(a_{\lambda\mu})$ of size $(\rho_j - \rho_{j-1}) \times (\rho_{s+1} - \rho_s)$. In order to compute $d_\rho = \dim \mathbb{F}_\rho(V)$, we have to multiply the number of rows and columns of each upper block of zeros in $(a_{\lambda\mu})$ and then sum all these products together. Hence,

$$\begin{aligned}
d_\rho &= \sum_{j=1}^{m-1} \left[(\rho_j - \rho_{j-1}) \left(\sum_{s=j}^{m-1} (\rho_{s+1} - \rho_s) \right) \right] \\
&= \sum_{j=1}^{m} [(\rho_j - \rho_{j-1})(r - \rho_j)] \\
&= r \sum_{j=1}^{m} \rho_j - r \sum_{j=1}^{m} \rho_{j-1} - \sum_{j=1}^{m} (\rho_j^2 - \rho_j \rho_{j-1}) \\
&= r^2 + r \left(\sum_{j=1}^{m-1} \rho_j - \sum_{j=1}^{m} \rho_{j-1} \right) - \frac{1}{2} \left(r^2 + \sum_{j=1}^{m} (\rho_j - \rho_{j-1})^2 \right) \\
&= r^2 - r\rho_0 - \frac{r^2}{2} - \frac{1}{2} \sum_{j=1}^{m} (\rho_j - \rho_{j-1})^2
\end{aligned}$$

and we get Formula (1.2). Formula (1.3) follows immediately. □

1.2.1 Universal vector bundles

The tautological filtration (1.1) gives $\binom{m+1}{2}$ *universal* vector bundles over $\mathbb{F}_\rho(E)$. These are of course tautological vector bundles already introduced (we are not considering here the vector bundle of rank 0), and all the possible quotients of the form $U_{\rho,l}/U_{\rho,\ell}$, for each $0 < \ell < l \leq m$.

Example 1.3. Set $m = 2$. If ρ is the sequence $(0, 1, r)$ then $\mathbb{F}_\rho(E)$ equals the projective bundle of lines in E, which we denote by $\mathbb{P}(E)$. In this case, the tautological filtration (1.1) consists on one proper sub-bundle only, namely $U_{(0,1,r),1}$, which equals, by definition, the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$. Given that $U_{\rho,2}/U_{\rho,1}$ is the only quotient given by the filtration (1.1), we have that $\mathbb{P}(E)$ has three universal vector bundles. They form the tautological short exact sequence

$$0 \to \underbrace{\mathcal{O}_{\mathbb{P}(E)}(-1)}_{U_{\rho,1}} \hookrightarrow \underbrace{\pi_\rho^* E}_{U_{\rho,2}} \twoheadrightarrow \underbrace{\pi_\rho^* E / \mathcal{O}_{\mathbb{P}(E)}(-1)}_{U_{\rho,2}/U_{\rho,1}} \to 0 \tag{1.4}$$

over the projective bundle $\mathbb{P}(E)$.

Remark 1.4. Suppose that E is equipped with a Hermitian metric h. The pull-back metric $\pi_\rho^* h$ on $\pi_\rho^* E$ endows the vector bundle $U_{\rho,l}$ with the restriction metric $h_{\rho,l} := \pi_\rho^* h\big|_{U_{\rho,l}}$. Consequently, the quotient bundle $U_{\rho,l}/U_{\rho,\ell}$ is equipped with the Hermitian metric given by the quotient of the metrics $h_{\rho,l}$ and $h_{\rho,\ell}$. When needed, we denote such a Hermitian metric by $H_{\rho,(l,\ell)}$.

Definition 1.5. Let $\rho = (\rho_0, \dots, \rho_m)$ and $\tau = (\tau_0, \dots, \tau_{\tilde{m}})$ be two sequences of the form $(\star)$. We say that ρ is *greater than or equal to* τ, and we write $\rho \geq \tau$, if:

(a) $m \geq \tilde{m}$;

(b) for each $\tilde{l} \in \{0, \dots, \tilde{m}\}$ there is a index $l \in \{0, \dots, m\}$ for which $\rho_l = \tau_{\tilde{l}}$.

In addition, if

(c) $\rho \neq \tau$;

we write $\rho > \tau$.

Given $\rho > \tau$, it is useful to denote by

$$\pi_\tau^\rho \colon \mathbb{F}_\rho(E) \to \mathbb{F}_\tau(E)$$

the natural forgetful projection between flag bundles. If $\rho = \tau$, then π_ρ^ρ is simply the identity map of $\mathbb{F}_\rho(E)$.

By construction, $\pi_\tau \circ \pi_\tau^\rho = \pi_\rho$, *i.e.*, the diagram

$$\begin{array}{ccc} \mathbb{F}_\rho(E) & \xrightarrow{\pi_\tau^\rho} & \mathbb{F}_\tau(E) \\ & {\scriptstyle \pi_\rho} \searrow \quad \swarrow {\scriptstyle \pi_\tau} & \\ & X & \end{array} \tag{1.5}$$

is commutative.

Moreover, if $\rho \geq \tau \geq \sigma$ then $\pi_\sigma^\tau \circ \pi_\tau^\rho = \pi_\sigma^\rho$, *i.e.*, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{F}_\rho(E) & \xrightarrow{\pi_\tau^\rho} & \mathbb{F}_\tau(E) \\ & {\scriptstyle \pi_\sigma^\rho} \searrow \quad \swarrow {\scriptstyle \pi_\sigma^\tau} & \\ & \mathbb{F}_\sigma(E) & \end{array}$$

of projections between flag bundles over X.

Remark 1.6. Given the projection $\pi^\rho_\tau \colon \mathbb{F}_\rho(E) \to \mathbb{F}_\tau(E)$, suppose that there is a index $\tilde{l}$ such that $\rho_l = \tau_{\tilde{l}}$. Then, it is straightforward to note that the tautological vector bundle $U_{\rho,l} \to \mathbb{F}_\rho(E)$ is the pull-back through π^ρ_τ of the bundle $U_{\tau,\tilde{l}} \to \mathbb{F}_\tau(E)$. Moreover, it is clear that the metric $h_{\rho,l}$ coincides with the pull-back metric $(\pi^\rho_\tau)^*(h_{\tau,\tilde{l}})$. Indeed,

$$(\pi^\rho_\tau)^*(h_{\tau,\tilde{l}}) = (\pi^\rho_\tau)^* \left(\pi^*_\tau h|_{U_{\tau,\tilde{l}}}\right) = (\pi^\rho_\tau)^*\pi^*_\tau h|_{(\pi^\rho_\tau)^* U_{\tau,\tilde{l}}} = \pi^*_\rho h|_{U_{\rho,l}} = h_{\rho,l}.$$

In addition, for $0 < \ell < l$, if there is a $\tilde{\ell}$ such that $\rho_\ell = \tau_{\tilde{\ell}}$, then $U_{\rho,l}/U_{\rho,\ell} \to \mathbb{F}_\rho(E)$ is the pull-back through π^ρ_τ of the quotient $U_{\tau,\tilde{l}}/U_{\rho,\tilde{\ell}} \to \mathbb{F}_\tau(E)$. In this case, the Hermitian metric $H_{\rho,(l,\ell)}$ is the pull-back through π^ρ_τ of the metric $H_{\tau,(\tilde{l},\tilde{\ell})}$.

In particular, we emphasize the following.

Remark 1.7. Choose $m \geq 2$, and let ρ be as in $(\star)$. Fix an index $l = 1, \dots, m-1$ and set $s := \rho_l$. Since $\rho \geq (0, s, r)$, we have of course the forgetful projection

$$\pi^\rho_s \colon \mathbb{F}_\rho(E) \to \mathbb{G}_s(E)$$

on the Grassmannian bundle $\mathbb{G}_s(E)$ of s-planes in E.

By Remark 1.6, $U_{\rho,l}$ is the pull-back through π^ρ_s of the tautological s-plane bundle over $\mathbb{G}_s(E)$ denoted by γ_s. Moreover, the Hermitian metric $h_{\rho,l}$ is the pull-back of the obvious restriction metric h_s on γ_s induced by h. Similarly, $(\pi^*_\rho E/U_{\rho,l}, H_{\rho,(m,l)})$ is the pull-back through π^ρ_s of the universal quotient bundle Q_s over $\mathbb{G}_s(E)$, equipped with the quotient metric H_s.

Finally, if $m > 2$ and $0 < \ell < l < m$, let $\mathbf{s}$ be the sequence $(0, s_1, s_2, r)$ with $s_1 = \rho_\ell$, $s_2 = \rho_l$. Given that $\rho \geq \mathbf{s}$, we have that $U_{\rho,l}/U_{\rho,\ell}$ is the pull-back through the map $\pi^\rho_{\mathbf{s}}$ of the unique quotient $Q_{\mathbf{s}} := U_{\mathbf{s},2}/U_{\mathbf{s},1}$ of proper sub-bundles over the flag bundle $\mathbb{F}_{\mathbf{s}}(E)$. As always, $Q_{\mathbf{s}}$ is equipped with the obvious quotient metric which we denote by $H_{\mathbf{s}}$, and $(\pi^\rho_{\mathbf{s}})^* H_{\mathbf{s}} = H_{\rho,(l,\ell)}$.

The observations in Remark 1.7 are the starting point for the proof of Theorem 2.11.

Local frames of universal vector bundles

Fix a point $x_0 \in X$ and local holomorphic coordinates z centered at x_0. Let ρ be a sequence as in $(\star)$ and fix a flag $\mathbf{f}_0 \in \mathbb{F}_\rho(E_{x_0})$.

Given a local normal frame $(e_1, \dots, e_r)$ of E at x_0, by [Dem88a] we can always suppose that the flag

$$\{0_{x_0}\} \subset \cdots \subset \operatorname{Span}\{e_{r-\rho_l+1}(x_0), e_{r-\rho_l+2}(x_0), \dots, e_r(x_0)\} \subset \cdots \subset E_{x_0}$$

of $\mathbb{F}_\rho(E_{x_0})$ coincides with $\mathbf{f}_0$.

The basis $(e_1(z), \dots, e_r(z))$ gives affine coordinates $\zeta = (\zeta_{\lambda\mu})$ on the fiber $\mathbb{F}_\rho(E_z)$, cf. with the proof of Proposition 1.2, where the indices λ and μ satisfy the condition

$$\text{there exists } \ell = 1, \dots, m-1 \text{ for which } 1 \leq \lambda \leq r - \rho_{m-\ell} < \mu \leq r. \qquad (*)$$

Such coordinates parameterize flags of E_z of type

$$\{0_z\} \subset \cdots \subset V_{z,\rho_l} := \operatorname{Span}\{\epsilon_{r-\rho_l+1}(z,\zeta), \epsilon_{r-\rho_l+2}(z,\zeta), \dots, \epsilon_r(z,\zeta)\} \subset \cdots \subset E_z$$

where, for $1 \leq \alpha \leq r$,

$$\epsilon_\alpha(z,\zeta) = e_\alpha(z) + \sum \zeta_{\lambda\alpha}\, e_\lambda(z)$$

and the summation is taken over all $1 \leq \lambda < \alpha$ such that, as before, the indices λ and α satisfy Condition $(*)$. Summing up, we have constructed in this way local holomorphic coordinates $(z,\zeta) = (z_1, \ldots, z_n, \zeta_{\lambda\mu})$ on $\mathbb{F}_\rho(E)$ centered at $(x_0, \mathbf{f}_0)$.

By construction, if $0 < l < m$, the local sections

$$\epsilon_\alpha(z,\zeta), \quad r - \rho_l < \alpha \leq r, \tag{1.6}$$

form a local holomorphic frame of $U_{\rho,l}$. Moreover, a local holomorphic frame for $\pi_\rho^* E/U_{\rho,l}$ is given by the sections

$$\tilde{e}_\alpha(z,\zeta) = \text{image of } e_\alpha(z) \text{ in } {E_z}\big/{V_{z,\rho_l}}, \quad 1 \leq \alpha \leq r - \rho_l. \tag{1.7}$$

Similarly, if $0 < \ell < l < m$, the sections

$$\tilde{\epsilon}_\alpha(z,\zeta) = \text{image of } \epsilon_\alpha(z,\zeta) \text{ in } {V_{z,\rho_l}}\big/{V_{z,\rho_\ell}}, \quad r - \rho_l < \alpha \leq r - \rho_\ell, \tag{1.8}$$

form a local holomorphic frame for $U_{\rho,l}/U_{\rho,\ell}$.

In Section 2.3.1 we use the local frames (1.6), (1.7), (1.8) to compute the Chern curvature tensors in a point of all the $\binom{m+1}{2}$ universal vector bundles of $\mathbb{F}_\rho(E)$.

1.2.2 Universal line bundles and their curvature

Over the flag bundle $\mathbb{F}_\rho(E)$ one can define canonical line bundles as follows: for $1 \leq j \leq m$ set

$$Q_{\rho,j} := \det(U_{\rho,m-j+1}/U_{\rho,m-j}).$$

For any multi-index $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ satisfying

$$a_{r-\rho_{m-j+1}+1} = a_{r-\rho_{m-j+1}+2} = \cdots = a_{r-\rho_{m-j}}, \; 1 \leq j \leq m, \tag{1.9}$$

set, for $0 \leq j \leq m$, $s_j := r - \rho_{m-j}$ and define

$$Q_\rho^{\mathbf{a}} := {Q_{\rho,1}}^{\otimes a_{s_1}} \otimes \cdots \otimes {Q_{\rho,m}}^{\otimes a_{s_m}}.$$

Example 1.8. In the notation of Example 1.3, consider the projective bundle $\mathbb{P}(E)$ on X. By taking the determinant of the short exact sequence (1.4), we have that

$$Q_{\rho,1} = \det\left(\pi_\rho^* E/\mathcal{O}_{\mathbb{P}(E)}(-1)\right) \cong \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi_\rho^* \det E$$

and

$$Q_{\rho,2} = \mathcal{O}_{\mathbb{P}(E)}(-1).$$

Therefore,

$$Q_\rho^{(a_1,a_2)} = \mathcal{O}_{\mathbb{P}(E)}(a_1 - a_2) \otimes \left(\pi_\rho^* \det E\right)^{\otimes a_1}.$$

If E is equipped with a Hermitian metric h, then we have an induced metric on all the line bundles $Q_{\rho,j}$'s. For $j < m$ (resp. $j = m$), this metric is the determinant of the quotient metric $H_{\rho,(m-j+1,m-j)}$ (resp. of the restriction metric $h_{\rho,1}$) mentioned in Remark 1.4. Consequently, we have natural metrics induced on all the line bundles $Q_\rho^{\mathbf{a}}$. In order to simplify the notation, it is useful to denote by h all these mentioned metrics by a slight abuse.

For a given multi-index $\mathbf{a} \in \mathbb{Z}^r$ satisfying Condition (1.9), in [Dem88a, Formula (4.9)] the Chern curvature $\Theta(Q_\rho^{\mathbf{a}}, h)$ of $(Q_\rho^{\mathbf{a}}, h)$ at the point $(x_0, \mathbf{f}_0)$ is computed and reads

$$\Theta(Q_\rho^{\mathbf{a}}, h)_{(x_0,\mathbf{f}_0)} = \sum_\lambda a_\lambda \Theta_{\lambda\lambda}(x_0) + \sum_{\lambda,\mu}(a_\lambda - a_\mu) d\zeta_{\lambda\mu} \wedge d\bar{\zeta}_{\lambda\mu}, \tag{1.10}$$

where λ and μ in the second summation satisfy Condition $(*)$, in order to have that the $\zeta_{\lambda\mu}$'s are defined. Let

$$\Xi_\rho^{\mathbf{a}} := c_1(Q_\rho^{\mathbf{a}}, h) = \frac{i}{2\pi}\Theta(Q_\rho^{\mathbf{a}}, h) \in \mathcal{A}^{1,1}(\mathbb{F}_\rho(E))$$

be the first Chern form of $(Q_\rho^{\mathbf{a}}, h)$, which represents the first Chern class $c_1(Q_\rho^{\mathbf{a}})$.

Remark 1.9. In the particular case where the multi-index is non increasing, which is indeed the case actually considered in [Dem88a, Formula (4.9)], and which shall be of special interest later, Condition $(*)$ on λ and μ in the second summation above is equivalent to require $a_\lambda > a_\mu$, so that the curvature formula becomes

$$\Theta(Q_\rho^{\mathbf{a}}, h)_{(x_0,\mathbf{f}_0)} = \sum_\lambda a_\lambda \Theta_{\lambda\lambda}(x_0) + \sum_{a_\lambda > a_\mu}(a_\lambda - a_\mu) d\zeta_{\lambda\mu} \wedge d\bar{\zeta}_{\lambda\mu}. \tag{1.11}$$

Observe that $\sum_{a_\lambda > a_\mu}(a_\lambda - a_\mu) d\zeta_{\lambda\mu} \wedge d\bar{\zeta}_{\lambda\mu}$ gives a positive definite block for the curvature of $(Q_\rho^{\mathbf{a}}, h)$ if and only if the non increasing multi-index $\mathbf{a}$ is strictly decreasing at each place where it is allowed by Condition (1.9), *i.e.*

$$a_{s_1} > a_{s_2} > \cdots > a_{s_m}. \tag{1.12}$$

For $1 \le j \le m$, denote by $\mathbf{1}_j$ the element of $\mathbb{Z}^r$ with 1's in the places $s_{j-1} + 1, s_{j-1} + 2, \dots, s_j$, and 0's elsewhere. By definition, we have that $Q_\rho^{\mathbf{1}_j} = Q_{\rho,j}$, and thanks to Formula (1.10) we can recover the curvature of the line bundle $Q_{\rho,j}$ at the point $(x_0, \mathbf{f}_0) \in \mathbb{F}_\rho(E)$:

$$\begin{aligned}\Theta(Q_{\rho,j}, h)_{(x_0,\mathbf{f}_0)} = &\sum_{\lambda=s_{j-1}+1}^{s_j} \Theta_{\lambda\lambda}(x_0) \\ &- \sum_{\substack{\lambda=1,\dots,s_{j-1}\\ \mu=s_{j-1}+1,\dots,s_j}} d\zeta_{\lambda\mu} \wedge d\bar{\zeta}_{\lambda\mu} + \sum_{\substack{\lambda=s_{j-1}+1,\dots,s_j\\ \mu=s_j+1,\dots,r}} d\zeta_{\lambda\mu} \wedge d\bar{\zeta}_{\lambda\mu}.\end{aligned} \tag{1.13}$$

Finally, let

$$\Xi_{\rho,j} := c_1(Q_{\rho,j}, h) = \frac{i}{2\pi}\Theta(Q_{\rho,j}, h) \in \mathcal{A}^{1,1}(\mathbb{F}_\rho(E))$$

be the first Chern form of $(Q_{\rho,j}, h)$, which represents the first Chern class $c_1(Q_{\rho,j})$.

As before, we shall drop the subscript ρ and write Q_j, Ξ_j and $Q^{\mathbf{a}}$, $\Xi^{\mathbf{a}}$ in case of complete flag bundles.

1.2.3 Splitting of the tangent bundle

Consider the short exact sequence

$$0 \to \ker(d\pi_\rho) \hookrightarrow T_{\mathbb{F}_\rho(E)} \xrightarrow{d\pi_\rho} \pi_\rho^* T_X \to 0$$

induced by the differential of $\pi_\rho \colon \mathbb{F}_\rho(E) \to X$, where $T_{\mathbb{F}_\rho(E)}$ and T_X are the tangent bundles of $\mathbb{F}_\rho(E)$ and of X respectively. Recall that $\ker(d\pi_\rho)$ is the *relative tangent bundle*, usually denoted by $T_{\mathbb{F}_\rho(E)/X}$.

We now define a natural orthogonal splitting of $T_{\mathbb{F}_\rho(E)}$ into a vertical and horizontal part. In order to do this, observe that for any given weight $\mathbf{a}$ as described at the end of Remark 1.9, by Formula (1.11) the line bundle $Q_\rho^{\mathbf{a}}$ is relatively positive. Hence, $\Xi_\rho^{\mathbf{a}}$ gives a positive definite Hermitian form whenever restricted to the relative tangent bundle $T_{\mathbb{F}_\rho(E)/X}$. Therefore, for any such $\mathbf{a}$, we get a corresponding orthogonal decomposition (in the smooth category)

$$T_{\mathbb{F}_\rho(E)} = T_{\mathbb{F}_\rho(E)/X} \oplus T^{\perp_{\Xi_\rho^{\mathbf{a}}}}_{\mathbb{F}_\rho(E)/X}.$$

Finally, from the explicit expression of Formula (1.11), we see that such a decomposition is independent of the particular choice of the weight $\mathbf{a}$, so that it depends only on h and we can drop any reference to the weight and write

$$T_{\mathbb{F}_\rho(E)} = T_{\mathbb{F}_\rho(E)/X} \oplus T^{\perp_h}_{\mathbb{F}_\rho(E)/X}. \tag{1.14}$$

We denote by

$$p_1 \colon T_{\mathbb{F}_\rho(E)} \to T_{\mathbb{F}_\rho(E)/X} \quad \text{and} \quad p_2 \colon T_{\mathbb{F}_\rho(E)} \to T^{\perp_h}_{\mathbb{F}_\rho(E)/X}$$

the natural projections relative to the splitting (1.14).

Remark 1.10. Observe that the above splitting is compatible with $d\pi_\rho$ in the following sense. The restriction $d\pi_\rho|_{T^{\perp_h}_{\mathbb{F}_\rho(E)/X}} \colon T^{\perp_h}_{\mathbb{F}_\rho(E)/X} \to \pi_\rho^* T_X$ is a smooth isomorphism of complex vector bundles, and moreover, by a direct pointwise computation in the local holomorphic coordinates chosen above, we have $d\pi_\rho = d\pi_\rho \circ p_2$ and $d\pi_\rho \circ p_1 = 0$.

In particular, observe that, for any given point $(x_0, \mathbf{f}_0) \in \mathbb{F}_\rho(E)$, given the holomorphic coordinates (z, ζ) centered at $(x_0, \mathbf{f}_0)$ as above, we explicitly have that

$$T_{\mathbb{F}_\rho(E)/X,(x_0,\mathbf{f}_0)} = \operatorname{Span}\left\{ \left.\frac{\partial}{\partial \zeta_{\lambda\mu}}\right|_{(z,\zeta)=(0,0)} \right\},$$
$$T^{\perp_h}_{\mathbb{F}_\rho(E)/X,(x_0,\mathbf{f}_0)} = \operatorname{Span}\left\{ \left.\frac{\partial}{\partial z_k}\right|_{(z,\zeta)=(0,0)} \right\}.$$

1.3 Schur polynomials

We recall here some notation about Schur polynomials, essentially taken from the exposition in [FL83] (see also [DPS94, § 2] and [Ful97]).

Denote by $\Lambda(k, r)$ the set of all the partitions $\sigma = (\sigma_1, \ldots, \sigma_k)$ in $\mathbb{N}^k$ such that

$$r \geq \sigma_1 \geq \cdots \geq \sigma_k \geq 0, \quad |\sigma| = \sum_{j=1}^{k} \sigma_j = k.$$

For every $\sigma \in \Lambda(k,r)$ we can define a *Schur polynomial* $S_\sigma \in \mathbb{Z}[c_1,\dots,c_r]$ of weighted degree $2k$ (we regard c_j as having degree $2j$) as

$$\begin{aligned} S_\sigma(c_1,\dots,c_r) &:= \det\bigl(c_{\sigma_i+j-i}\bigr)_{1\le i,j\le k} \\ &= \det\begin{pmatrix} c_{\sigma_1} & c_{\sigma_1+1} & \cdots & c_{\sigma_1+k-1} \\ c_{\sigma_2-1} & c_{\sigma_2} & \cdots & c_{\sigma_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\sigma_k-k+1} & c_{\sigma_k-k+2} & \cdots & c_{\sigma_k} \end{pmatrix} \end{aligned}$$

where, by convention, $c_0 = 1$ and $c_\ell = 0$ if $\ell \notin [0,r]$.

The Schur polynomials, as $\sigma \in \Lambda(k,r)$ varies, form a basis for the $\mathbb{Q}$-vector space of degree $2k$ weighted homogeneous polynomials in r variables. Thus, given such a polynomial P we can write

$$P = \sum_{\sigma\in\Lambda(k,r)} b_\sigma(P) S_\sigma.$$

The set of all P such that $b_\sigma(P) \ge 0$ for every $\sigma \in \Lambda(k,r)$, which is called the set of *positive polynomials*, is of course a positive convex cone, which we call $\Pi(r)$ following [Gri69] (remark that this is not exactly the positive cone considered by Griffiths, but they coincides *a posteriori* thanks to the work of [FL83]). It is well known that any product of Schur polynomials can be written as a linear combination of Schur polynomials with non-negative integral coefficients; the values of these coefficients is given combinatorially by the Littlewood–Richardson rule. Thus, these positive cones are stable under product (cf. with the analogous property for wedge product of strongly positive forms, as observed in Remark 3.7).

Now, if E is a rank r holomorphic vector bundle over a complex manifold X and if σ is a partition in $\Lambda(k,r)$, the *Schur class* of E associated to σ is the cohomology class

$$S_\sigma(E) := S_\sigma(c_1(E),\dots,c_r(E)) \in H^{2k}(X,\mathbb{Z}) \tag{1.15}$$

formally obtained by computing S_σ on the the Chern classes of E.

In the same way, if h is a Hermitian metric on E we can define the *Schur form* of (E,h) associated to σ, formally obtained by computing S_σ on the the Chern forms of (E,h), and we denote it by $S_\sigma(E,h)$. Clearly, the closed differential $2k$-form $S_\sigma(E,h)$ is a special representative for the class $S_\sigma(E)$.

Example 1.11. Denote by $s_k(E,h)$ the k-th Segre form of (E,h), which represents the Segre class $s_k(E)$. Given a partition $\sigma \in \Lambda(k,r)$, interesting examples of Schur forms are:

$$S_\sigma(E,h) = \begin{cases} c_k(E,h) & \text{if } \sigma = (k,0,\dots,0), \\ c_1(E,h)\wedge c_{k-1}(E,h) - c_k(E,h) & \text{if } \sigma = (k-1,1,0,\dots,0), \\ (-1)^k s_k(E,h) & \text{if } \sigma = (1,\dots,1), \end{cases}$$

and at the level of cohomology classes we have similar equalities.

1.3.1 Generalized Schur classes and forms

Here we want to introduce a family of cohomology classes (resp. differential forms) which are given by dropping the assumption that σ is a partition in $\Lambda(k,r)$. The following notation is taken from [DP17, §4], and will be essential in order to state Formula (2.19).

Definition 1.12. Let $\sigma = (\sigma_1,\dots,\sigma_k) \in \mathbb{Z}^k$ be a sequence of integers. We define a cohomology class

$$s_\sigma(E) := \det\big(s_{\sigma_i+j-i}(E)\big)_{1\le i,j\le k}$$

in $H^{2|\sigma|}(X,\mathbb{Z})$, where, as usual, $s_0(E)=1$ and $s_\ell(E)=0$ if $\ell \notin [0,n]$.

The relationship between Schur classes and Definition 1.12 is given in the following.

Example 1.13. Let σ be a partition in $\Lambda(k,r)$. It follows from the Jacobi–Trudi identities ([Ful97]) that

$$s_\sigma(E) = (-1)^{|\sigma|} S_{\sigma'}(E), \tag{1.16}$$

where σ' is the conjugate partition of σ, obtained through the transposition of the Young diagram of σ. For instance, by Formula (1.16) the partition $(1,\dots,1)$ gives $(-1)^k c_k(E)$, which is the k-th signed Chern class, while the Segre class $s_k(E)$ is associated to the partition $(k,0,\dots,0)$.

Note that the sign $(-1)^{|\sigma|}$ in Formula (1.16) is due to the fact that the definition of Schur class given in Formula (1.15) involves Chern classes, while Definition 1.12 is given in terms of Segre classes. The reason why we choose the Segre classes' approach in Definition 1.12 is to make our notation uniform with that of [DP17]. This choice is very useful also in Section 3.2.1.

Similarly to Definition 1.12, we can associate to any $\sigma \in \mathbb{Z}^k$ the differential form

$$s_\sigma(E,h) := \det\big(s_{\sigma_i+j-i}(E,h)\big)_{1\le i,j\le k}.$$

Inspired by Formula (1.16), we call $s_\sigma(E,h)$ (resp. $s_\sigma(E)$) the *generalized Schur form* (resp. *class*) associated to σ.

1.4 Positivity notions

In this section we give some notions of positivity for differential forms and for vector bundles, needed to state the results in Chapter 3.

1.4.1 Positivity notions for differential forms

We recall here some basic notions about the positivity of differential forms. The following exposition is taken, mainly, from [HK74] and [Dem12, § III].

Let V be a complex vector space of dimension n and let $(e_1,\dots,e_n)$ be a basis of V. The notation $(e_1^\vee,\dots,e_n^\vee)$ stands for the corresponding dual basis of $V^\vee$. For $0\le p,q\le n$, denote by $\Lambda^{p,q}V^\vee$ the space of exterior forms of bi-degree (p,q) on V. A form $u\in\Lambda^{p,q}V^\vee$ is *real* if $\bar u = u$; since $\bar u \in \Lambda^{q,p}V^\vee$, it follows that $p=q$ in this case. Let $\Lambda^{p,p}_{\mathbb{R}}V^\vee$ denote the space of real (p,p)-forms.

Definition 1.14. A form $\nu \in \Lambda^{n,n}V^\vee$ is called a *positive* volume form if

$$\nu = \tau \; ie_1^\vee \wedge \bar{e}_1^\vee \wedge \cdots \wedge ie_n^\vee \wedge \bar{e}_n^\vee$$

for some $\tau \in \mathbb{R}$, $\tau \geq 0$.

Of course, this notion is independent on the choice of the basis $(e_1, \dots, e_n)$.

From now on, for $0 \leq p \leq n$ set $q = n - p$. Recall that a $(q, 0)$-form which can be expressed as $\beta_1 \wedge \cdots \wedge \beta_q$ for $\beta_1, \dots, \beta_q \in V^\vee$ is called *decomposable*.

Definition 1.15. An exterior form $u \in \Lambda^{p,p}_{\mathbb{R}}V^\vee$ is called

- *positive*, if for every $\beta \in \Lambda^{q,0}V^\vee$ decomposable, $u \wedge i^{q^2}\beta \wedge \bar{\beta}$ is a positive volume form;
- *Hermitian positive*, if for every $\beta \in \Lambda^{q,0}V^\vee$, $u \wedge i^{q^2}\beta \wedge \bar{\beta}$ is a positive volume form;
- *strongly positive*, if there are decomposable forms $\alpha_1, \dots, \alpha_N \in \Lambda^{p,0}V^\vee$ such that u can be expressed as $\sum_{s=1}^N i^{p^2}\alpha_s \wedge \bar{\alpha}_s$.

Remark 1.16. The terminology used for the positivity of forms is, currently, not always standard in the literature. We have chosen here to follow the terminology of [Dem12, §III], which is a standard reference for positivity of forms. Note however that [Dem12, §III] does not deal with the intermediate notion of positivity (which is called simply *positivity* in [HK74]). The reason why we call it *Hermitian* is due to the fact that $u \in \Lambda^{p,p}_{\mathbb{R}}V^\vee$ is Hermitian positive if and only if for each $\beta, \eta \in \Lambda^{q,0}V^\vee$, $(\beta, \eta) \mapsto u \wedge i^{q^2}\beta \wedge \bar{\eta}$ gives a positive semidefinite Hermitian form on $\Lambda^{q,0}V^\vee$.

Remark 1.17. Of course, with an appropriate modification of Definition 1.15, one can define the *strict* notions of positivity for forms. Mainly, we are dealing here with Griffiths semipositive vector bundles, so it is more natural to consider the three notions given in Definition 1.15, rather than the strict ones. Moreover, note that some authors (see, for instance, [Zhe00, Gul12, Li21]) use the term *non-negative*, or *semi-positive*, (resp. *positive*) instead of *positive* (resp. *strictly positive*).

Example 1.18. Let $\xi \in \Lambda^{p,0}V^\vee$, then $i^{p^2}\xi \wedge \bar{\xi}$ is a Hermitian positive form. Indeed, for every $\beta \in \Lambda^{q,0}V^\vee$, the wedge product $\xi \wedge \beta$ equals $\lambda e_1^\vee \wedge \cdots \wedge e_n^\vee$ for some $\lambda \in \mathbb{C}$, hence $i^{p^2}\xi \wedge \bar{\xi} \wedge i^{q^2}\beta \wedge \bar{\beta} = i^{n^2}\xi \wedge \beta \wedge \overline{\xi \wedge \beta}$ is a positive volume form.

Let $\mathrm{WP}^p\,V^\vee$, $\mathrm{HP}^p\,V^\vee$ and $\mathrm{SP}^p\,V^\vee$ denote respectively the closed positive convex cones contained in $\Lambda^{p,p}_{\mathbb{R}}V^\vee$ spanned by positive, Hermitian positive and strongly positive forms. The notations WP and SP are taken from [HK74]. It is straightforward to see that, in general,

$$\mathrm{SP}^p\,V^\vee \;\subseteq\; \mathrm{HP}^p\,V^\vee \;\subseteq\; \mathrm{WP}^p\,V^\vee. \tag{1.17}$$

Remark 1.19. The two inclusions in (1.17) become equalities when $p = 0, 1, n-1, n$. Indeed, if $p = 0, n$ all the positivity notions in Definition 1.15 do coincide. If $p = 1$, the one-to-one correspondence between Hermitian forms and real (1, 1)-forms shows by a diagonalization argument that every positive $(1, 1)$-form is strongly positive. By duality, it is also true if $p = n - 1$.

If K is a convex cone in $\Lambda^{q,q}_{\mathbb{R}} V^\vee$, then its *dual cone* is

$$\mathrm{K}^* := \{u \in \Lambda^{p,p}_{\mathbb{R}} V^\vee \mid u \wedge v \text{ is a positive volume form } \forall v \in \mathrm{K}\}.$$

By Definition 1.15, $\mathrm{WP}^p V^\vee = (\mathrm{SP}^q V^\vee)^*$ and given that the bidual of a convex cone is equal to its closure, we have that $\mathrm{SP}^p V^\vee = (\mathrm{WP}^q V^\vee)^*$. This leads to the following characterization of strong positivity.

Proposition 1.20. *A form $u \in \Lambda^{p,p}_{\mathbb{R}} V^\vee$ is strongly positive if and only if for every positive (q,q)-form v, $u \wedge v$ is a positive volume form.*

We also need the following characterization of Hermitian positivity, which follows from the more general [HK74, Theorem 1.2].

Proposition 1.21. *A form $u \in \Lambda^{p,p}_{\mathbb{R}} V^\vee$ is Hermitian positive if and only if there are $\xi_1, \dots, \xi_N \in \Lambda^{p,0} V^\vee$ such that $u = \sum_{s=1}^N i^{p^2} \xi_s \wedge \bar{\xi}_s$.*

From Proposition 1.21, we deduce that the wedge product of two Hermitian positive forms is again Hermitian positive (of course, an analogous property holds for strongly positive forms by definition). Moreover, it is now clear that the dual cone $(\mathrm{HP}^q V^\vee)^*$ equals $\mathrm{HP}^p V^\vee$.

We also want to recall that, sometimes, Proposition 1.21 is given in the literature as a definition; see for instance [Gri69, p. 240] and [Li21, §2].

Remark 1.22. If $2 \leq p \leq n-2$, the two inclusions in (1.17) are strict. To see this, it is sufficient to observe that the Hermitian positive form $i^{p^2} \xi \wedge \bar{\xi}$ is strongly positive if and only if ξ is decomposable (see [Dem12, §III, (1.10) Remark] and [HK74, Proposition 1.5]). Given that, for instance, the $(p,0)$-form $(e_1^\vee \wedge e_2^\vee + e_3^\vee \wedge e_4^\vee) \wedge e_5^\vee \wedge \cdots \wedge e_{p+2}^\vee$ is not decomposable, we deduce that $\mathrm{SP}^p V^\vee \subsetneq \mathrm{HP}^p V^\vee$. Since the duality of cones reverses the inclusions, we get also that $\mathrm{HP}^p V^\vee \subsetneq \mathrm{WP}^p V^\vee$.

An explicit example of a positive form which is not Hermitian positive can be found in [HK74, p. 50]. As a byproduct, [HK74] constructs a positive form and a Hermitian positive form, for which their wedge product is a negative volume form. Therefore, unlike $\mathrm{SP}^p V^\vee$ and $\mathrm{HP}^p V^\vee$, the cone of positive forms is not stable under wedge product (compare this with [BP13]).

It is useful to recall the following characterization of positivity.

Proposition 1.23. *A form $u \in \Lambda^{p,p}_{\mathbb{R}} V^\vee$ is positive if and only if, equivalently,*

1. *for every vector subspace $W \subseteq V$ with $\dim_{\mathbb{C}} W = p$, the restriction $u|_W$ is a positive volume form on W;*
2. *for every $w_1, \dots, w_p \in V$, $(-i)^{p^2} u(w_1, \dots, w_p, \overline{w}_1, \dots, \overline{w}_p) \geq 0$.*

We also recall the following fact (see [Dem12, § III, (1.12) Proposition]).

Proposition 1.24. *Let $A\colon \tilde{V} \to V$ be a complex linear map of complex vector spaces. If $u \in \Lambda^{p,p}_{\mathbb{R}} V^\vee$ is (Hermitian, resp. strongly) positive, then the pull-back $A^* u \in \Lambda^{p,p}_{\mathbb{R}} \tilde{V}^\vee$ is (Hermitian, resp. strongly) positive.*

Of course, all of the definitions and results given in this section can be extended to a complex manifold X. It is sufficient to take $V = T_{X,x}$ for every $x \in X$ and to check that all the concepts expressed here are independent by change of holomorphic coordinates. This follows from Definition 1.14.

Lastly, we recall a well-known property of positive forms on complex manifolds, for which we give the proof.

Proposition 1.25. *Let $\pi\colon S \to X$ be a proper holomorphic submersion of complex manifolds. If η is a (Hermitian, resp. strongly) positive form on S, then the push-forward $\pi_*\eta$ is a (Hermitian, resp. strongly) positive form on X.*

Proof. First of all, suppose that η is a positive top degree form on S, and denote by s the relative dimension of π. Let dV_X be a volume form on X and Ω be a positive $(1,1)$-form on S. By Remark 1.19 and Proposition 1.24, $\Omega^{\wedge s} \wedge \pi^* dV_X$ is a positive top degree form on S. Hence, there is a non-negative function $f \in C^\infty(S)$ such that

$$\eta = f\ \Omega^{\wedge s} \wedge \pi^* dV_X.$$

For every $x \in X$, it follows from the projection formula that

$$\pi_*\eta_x = \left(\pi_*\left(f\ \Omega^{\wedge s}\right) \wedge dV_X\right)_x = \left(\int_{y\in\pi^{-1}(x)} f(x,y)\ \Omega^{\wedge s}_{(x,y)}\right) dV_x.$$

The integral on the right hand side is non-negative since the function f is. Hence $\pi_*\eta$ is a positive top degree form on X.

Set $n = \dim X$. For $0 \le k < n$, suppose now that η is a (Hermitian, resp. strongly) positive $(k+s, k+s)$-form on S, and take any (Hermitian, resp. positive) strongly positive $(n-k, n-k)$-form α on X. By the previous point, Definition 1.15, Propositions 1.20 and 1.24, and the projection formula, we have that the top degree form

$$\pi_*\eta \wedge \alpha = \pi_*(\eta \wedge \pi^*\alpha)$$

is positive on X. Therefore $\pi_*\eta$ is a (Hermitian, resp. strongly) positive (k,k)-form on X. □

1.4.2 Positivity notions for vector bundles

A Hermitian holomorphic line bundle (L,h) over a complex manifold X is *positive* if the Chern curvature $i\Theta(L,h)$ is a positive $(1,1)$-form.

If X is compact, then a line bundle $\mathcal{L}$ over X is *ample* if there is a holomorphic embedding $\iota\colon X \hookrightarrow \mathbb{P}^N$ for which $\mathcal{L}^{\otimes k} \cong \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$ for some $k > 0$, hence X is projective.

Since the hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ equipped with the standard Fubini–Study metric is positive (its curvature is indeed the Fubini–Study Kähler form on $\mathbb{P}^N$), then $\mathcal{L}$ endowed with the k-th root of the pull-back metric is a positive line bundle. Conversely, any positive line bundle over a compact complex manifold is ample by the Kodaira embedding theorem [Kod54].

Vector bundles of higher rank have several notions of positivity. We refer, for instance, to [Nak55, BC65, Gri69, Laz04, Dem12, LSY13, Fin21] and the references therein for more details on the theory of positive vector bundles.

In particular, there are different notions of *positive* Hermitian metric which are in general not equivalent. In this thesis, we consider essentially one of these notions which we describe below.

Let $(E, h) \to X$ be a Hermitian holomorphic vector bundle of rank r over a complex manifold of dimension n. At any point $x \in X$ and with respect to a unitary frame $(e_1, \ldots, e_r)$ for E_x, write the Chern curvature as

$$\Theta(E, h)_x = \sum_{1 \leq \alpha, \beta \leq r} \sum_{1 \leq j, k \leq n} c_{jk\alpha\beta} \, dz_j \wedge d\bar{z}_k \otimes e_\alpha^\vee \otimes e_\beta.$$

To $i\Theta(E, h) \in \mathcal{A}^{1,1}(X, \operatorname{Herm}(E, h))$ we can associate a natural Hermitian form $\theta(E, h)$ on $T_X \otimes E$ which in $x \in X$ is defined as

$$\theta(E, h)_x = \sum_{1 \leq \alpha, \beta \leq r} \sum_{1 \leq j, k \leq n} c_{jk\alpha\beta} \, (dz_j \otimes e_\alpha^\vee) \otimes (\overline{dz_k \otimes e_\beta^\vee}).$$

Definition 1.26 ([Gri69])**.** The vector bundle (E, h) is *Griffiths semipositive* (resp. *Griffiths positive*) if for every $x \in X$, $v = \sum v_\alpha e_\alpha \in E_x$, $\tau = \sum \tau_j \frac{\partial}{\partial z_j} \in T_{X,x}$ we have

$$\langle \Theta(E, h)_x \cdot v, v \rangle_h (\tau, \bar{\tau}) = \sum_{1 \leq \alpha, \beta \leq r} \sum_{1 \leq j, k \leq n} c_{jk\alpha\beta} \, \tau_j \bar{\tau}_k v_\alpha \bar{v}_\beta \geq 0$$

(resp. > 0 and $= 0$ if and only if v or τ is the zero vector).

In other words, the quadratic form associated to $\theta(E, h)_x$ takes nonnegative (resp. positive) values on non zero tensors $\tau \otimes v \in T_{X,x} \otimes E_x$ for any x.

By requiring that $\theta(E, h)$ is positive (semi)definite as a Hermitian form on $T_X \otimes E$ we get the stronger notion of *Nakano (semi)positivity* for (E, h) (see [Nak55]). Of course, there are the corresponding notions of Griffiths/Nakano (semi)negativity.

Since the Chern curvature tensor of the dual bundle $E^\vee$ equipped with the dual metric h^{-1} is given by the opposite of the transpose of $\Theta(E, h)$, we also have the notion of *dual Nakano (semi)positivity* for (E, h), which asks that $(E^\vee, h^{-1})$ is Nakano (semi)negative. See, for instance, [Dem12, Fin21] for more details on (dual) Nakano positivity and for additional concepts of positivity of h, which interpolate between Griffiths (semi)positivity and (dual) Nakano (semi)positivity.

By definition, Nakano (semi)positivity implies Griffiths (semi)positivity. Moreover, by recalling that (E, h) is Griffiths (semi)positive if and only if $(E^\vee, h^{-1})$ is Griffiths (semi)negative, we easily obtain that dual Nakano (semi)positivity implies Griffiths (semi)positivity. However, it is known that the holomorphic tangent bundle $T_{\mathbb{P}^N}$ with the Fubini–Study Kähler form is dual Nakano positive and Nakano semipositive, but $T_{\mathbb{P}^N}$ does not admit a Nakano positive Hermitian metric for any $N > 1$. See [LSY13] for an example of a Nakano positive and dual Nakano semipositive metric on a vector bundle which does not admit any dual Nakano positive metric. Observe that when $\dim X = 1$ or $\operatorname{rk} E = 1$ Nakano and dual Nakano (semi)positivity coincide with Griffiths (semi)positivity.

Now we recall the corresponding notion of *ampleness* for vector bundles for which we refer to [Laz04].

Definition 1.27 ([Har66])**.** A holomorphic vector bundle $\mathcal{V}$ over X is called *ample* if the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1)$ is ample over $\mathbb{P}(\mathcal{V}^\vee)$.

Griffiths positivity is considered as a differential geometric counterpart of ampleness. This because both notions share similar properties. For instance, it is well-known that any quotient of a Griffiths (semi)positive (resp. ample) vector bundle is Griffiths (semi)positive (resp. ample). Also, the direct sum or tensor power of two Griffiths (semi)positive (resp. ample) vector bundles is Griffiths (semi)positive (resp. ample).

In particular, it is well-known ([Gri69, Dem12]) that a Griffiths positive vector bundle over a compact complex manifold is ample. In the notation of Section 1.2, we can see that this also follows from Formula 1.13 with $\rho = (0, r-1, r)$ and $j = 1$. Indeed, in this case we have $\mathbb{F}_\rho(E) = \mathbb{G}_{r-1}(E) \cong \mathbb{P}(E^\vee)$ and $U_{\rho,1} = \gamma_{r-1}$. Thus,

$$Q_{\rho,j} = \pi_\rho^* E/\gamma_{r-1} \cong \mathcal{O}_{\mathbb{P}(E^\vee)}(1)$$

and Formula 1.13 proves that $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ is a positive line bundle.

It is a conjecture due to Griffiths [Gri69] that ampleness and Griffiths positivity are indeed equivalent. However, this is still open in general beside the case when X is a curve (see [Ume73, CF90]).

It is natural to ask if certain conditions of positivity on the vector bundle induce, in turn, the positivity of objects that can be defined from it. This kind of positivity issues can be placed at both algebraic (*e.g.*, ampleness) and differential geometric level (*e.g.*, Griffiths positivity). Here we are interested in the latter, and, for instance (see the introduction and Chapter 3), one may ask if the Chern forms coming from a Hermitian vector bundle (E, h) with Griffiths (semi)positive curvature are positive. The answer is trivially affirmative in the case of the first Chern form. Indeed, by definition

$$c_1(E, h) = \frac{i}{2\pi} \operatorname{tr}_{\operatorname{End}(E)} \Theta(E, h)$$

and the diagonal entries of the matrix $i\Theta(E, h)$ are positive $(1, 1)$-forms, since h is a Griffiths (semi)positive metric.

However, the answer to this question for other (special combinations of) Chern forms is absolutely non trivial, and Chapter 3 is dedicated to the study of this positivity issue.

Chapter 2

Curvature of universal bundles and push-forward formulæ

The aim of this chapter is to give explicit formulæ to compute the push-forward of a polynomial in the Chern forms of universal vector bundles. In this way, we are able to provide a differential geometric version of the Gysin formulæ for flag bundles (see, for instance, [Dam73, Ilo78, KT15, DP17]), which compute the push-forwards at the level of cohomology.

In Section 2.1 we describe the explicit expressions of the Gysin formulæ for flag bundles which we use throughout this thesis. These are the Darondeau–Pragacz formulæ ([DP17]). In Section 2.2 we prove Theorem B, which is crucial in Chapter 3 due to its applications in positivity. In Section 2.3 we compute the curvature of the universal vector bundles over flag bundles. This is the main ingredient for the proof of Theorem A, which is the metric counterpart of the universal Gysin formulæ for flag bundles in cohomology. Finally, in Section 2.4 we give an alternative version of our push-forward formulæ by using monomials in the virtual Chern roots.

From now on, suppose that E is a rank r holomorphic vector bundle over a complex n-dimensional manifold X. Let $m > 1$ be a natural number, and fix a sequence of dimensions $\rho = (\rho_0, \dots, \rho_m)$ satisfying Condition ($\star$). Let

$$\pi_\rho \colon \mathbb{F}_\rho(E) \to X$$

be the associated flag bundle, and denote by d_ρ its relative dimension.

2.1 Gysin formulæ *à la* Darondeau–Pragacz

We recall here a formula by Darondeau and Pragacz given in [DP17, Proposition 1.2], which allows to compute the push-forward of a polynomial in the Chern roots of the vector bundle E at the level of cohomology (in fact at the level of Chow rings, but cohomology suffices for our purposes).

First of all, observe that, even if [DP17, Proposition 1.2] is stated for algebraic manifolds over an algebraically closed field $\mathbb{K}$, when $\mathbb{K} = \mathbb{C}$ it also holds for non necessary algebraic complex manifolds. Indeed, ultimately, their proof only rely upon the push-forward formula for the tautological class on the projectivized bundle in terms of Segre classes, which is definitely valid also for general complex manifolds.

In what follows, for $k \in \mathbb{N}$, we use the same symbol for the push-forward

$$(\pi_\rho)_* \colon \mathcal{A}^{2(d_\rho+k)}(\mathbb{F}_\rho(E)) \to \mathcal{A}^{2k}(X)$$

of differential forms and the one induced in cohomology, namely

$$(\pi_\rho)_* \colon H^{2(d_\rho+k)}(\mathbb{F}_\rho(E)) \to H^{2k}(X).$$

With the same notation of [DP17], we denote by $\xi_1, \ldots, \xi_r$ the (virtual) Chern roots of $\pi_\rho^* E^\vee$. The Darondeau–Pragacz formula allows us to compute the push-forward of any cohomology class of the form

$$\tilde{F}(\xi_1, \ldots, \xi_r) \in H^{2(d_\rho+k)}(\mathbb{F}_\rho(E))$$

in terms of the Segre classes $s_j := s_j(E), 1 \le j \le n$, of the vector bundle E (we are thus implicitly asking here that the polynomial $\tilde{F}$ has the appropriate symmetries, which ensure that $\tilde{F}(\xi_\bullet)$ is a cohomology class).

More precisely, let $t_1, \ldots, t_r$ be a set of formal variables, then

$$\begin{aligned}(\pi_\rho)_* &\tilde{F}(\xi_1, \ldots, \xi_r) \\ &= [t_1^{b_1} \ldots t_r^{b_r}] \left(\tilde{F}(t_1, \ldots, t_r) \prod_{1 \le i \le r} \left(1 + \sum_{j=1}^{n} \frac{s_j}{t_i^j}\right) \prod_{1 \le i < j \le r} (t_i - t_j) \right)\end{aligned} \tag{2.1}$$

where, for a monomial $\mathfrak{m}$ and a (Laurent) polynomial P, the notation $[\mathfrak{m}](P)$ stands for the coefficient of $\mathfrak{m}$ in P, and the rule to determine the b_j's is as follows: for $r - \rho_\ell < j \le r - \rho_{\ell-1}$, say $j = r - \rho_\ell + i$ for some $i = 1, \ldots, \rho_\ell - \rho_{\ell-1}$, we set $b_j = r - i$.

Formula (2.1) is (one possible instance of) Darondeau–Pragacz formula.

Remark 2.1. Let us call $\Psi(s_1, \ldots, s_n)$ the right-hand side of Formula (2.1). If we consider the Segre classes $s_1, \ldots, s_n$ as formal variables, we can affirm that the polynomial $\Psi(s_1, \ldots, s_n)$ is universal, in the sense that its coefficients depends only upon $\tilde{F}$ and the rank r. Moreover, Ψ is, by construction, weighted homogeneous of degree $2k$, since $\deg s_j = 2j$.

In the notation of Section 1.3.1, we now recall the determinantal version of Darondeau–Pragacz formula stated in [DP17, Proposition 4.2]. We give it with a modification as follows.

Proposition 2.2. *Take the polynomial $\tilde{F}$ as before, and write*

$$\tilde{F}(\xi_1, \ldots, \xi_r) = \sum_{|\lambda| = d_\rho + k} a_\lambda \xi_1^{\lambda_1} \ldots \xi_r^{\lambda_r}.$$

If ν is the increasing sequence of integers determined by ρ as:

$$\nu_i = r - \rho_\ell \quad \text{for } r - \rho_\ell < i \le r - \rho_{\ell-1} \tag{2.2}$$

(in particular, if $m = r$ the sequence ν is given by $\nu_i = i - 1$), then, in terms of generalized Schur classes, we have

$$(\pi_\rho)_* \tilde{F}(\xi_1, \ldots, \xi_r) = \sum_{|\lambda| = d_\rho + k} a_\lambda s_{(\lambda - \nu)^\leftarrow}(E) \tag{2.3}$$

where the notation $(\sigma_1, \ldots, \sigma_r)^\leftarrow$ stands for $(\sigma_r, \ldots, \sigma_1)$ and the difference of λ and ν is defined componentwise.

Remark 2.3. Proposition 2.2 is stated differently from [DP17, Proposition 4.2]. Indeed, the original statement of Darondeau and Pragacz is given in terms of the Chern roots of $U^{\vee}_{\rho,m-1}$ (which is the greatest proper tautological sub-bundle of $\pi_\rho^* E^{\vee}$), while we need a push-forward formula involving all the Chern roots of $\pi_\rho^* E^{\vee}$.

What follows is the proof of Proposition 2.2, which is based on [DP17, Proposition 4.2] and [DP17, Proposition 1.2].

Proof of Proposition 2.2. Denote by $s_{1/t}$ the Laurent polynomial $1 + \sum_{j=1}^{n} s_j/t^j$. From Formula (2.1) we have

$$(\pi_\rho)_* \tilde{F}(\xi_1, \dots, \xi_r) = [t_1^{b_1} \cdots t_r^{b_r}] \left(\sum a_\lambda t_1^{\lambda_1} \cdots t_r^{\lambda_r} \prod_{1 \le i \le r} s_{1/t_i} \prod_{1 \le i < j \le r} (t_i - t_j) \right),$$

where $b_j = 2r - \rho_\ell - j$, for $r - \rho_\ell < j \le r - \rho_{\ell-1}$.

By the Vandermonde formula

$$\prod_{1 \le i < j \le r} (t_i - t_j) = \det\big(t_j^{r-i}\big)_{1 \le i,j \le r}$$

we have the following chain of equalities

$$\begin{aligned}
&(\pi_\rho)_* \tilde{F}(\xi_1, \dots, \xi_r) \\
&= \sum a_\lambda [t_1^{b_1} \cdots t_r^{b_r}] \left(t_1^{\lambda_1} \cdots t_r^{\lambda_r} \prod_{1 \le i \le r} s_{1/t_i} \det\big(t_j^{r-i}\big)_{1 \le i,j \le r} \right) \\
&= \sum a_\lambda [t_1^{b_1} \cdots t_r^{b_r}] \left(\det \left(t_j^{\lambda_j + r - i} s_{1/t_j} \right)_{1 \le i,j \le r} \right).
\end{aligned}$$

Moreover, for $M_{ij} \in H^\bullet(X)[t_j, t_j^{-1}]$ where $1 \le i,j \le r$, the identity

$$[t_1^{b_1} \cdots t_r^{b_r}]\big(\det(M_{ij})\big) = \det\left([t_j^{b_j}](M_{ij}) \right)$$

(see [DP17, Lemma 4.1]) gives

$$\begin{aligned}
&(\pi_\rho)_* \tilde{F}(\xi_1, \dots, \xi_r) \\
&= \sum a_\lambda \det \left([t_j^{b_j}] \left(t_j^{\lambda_j + r - i} s_{1/t_j} \right) \right)_{1 \le i,j \le r} \\
&= \sum a_\lambda \det \left(s_{\lambda_j + r - i - b_j} \right)_{1 \le i,j \le r}.
\end{aligned}$$

Since $s_{\lambda_j + r - i - b_j} = s_{\lambda_j - (r - \rho_\ell) + j - i} = s_{\lambda_j - \nu_j + j - i}$, we obtain a generalized Schur class after the following operations

$$\begin{aligned}
\left(s_{\lambda_j - \nu_j + j - i} \right) &\mapsto \left(s_{\lambda_i - \nu_i + i - j} \right) && \text{by } i \leftrightarrow j \\
&\mapsto \left(s_{\lambda_{r-i+1} - \nu_{r-i+1} + (r-i+1) - j} \right) && \text{by } (r - i + 1) \to i \\
&\mapsto \left(s_{\lambda_{r-i+1} - \nu_{r-i+1} + (r-i+1) - (r-j+1)} \right) && \text{by } (r - j + 1) \to j \\
&\mapsto \left(s_{\lambda_{r-i+1} - \nu_{r-i+1} + j - i} \right)
\end{aligned}$$

and we have

$$(\pi_\rho)_* \tilde{F}(\xi_1, \dots, \xi_r) = \sum_{|\lambda|=d+k} a_\lambda \det \left(s_{\lambda_{r-i+1}-\nu_{r-i+1}+j-i}\right)_{1\le i,j\le r}$$

which is Formula (2.3). □

2.2 Push-forward formulæ for universal line bundles

Suppose that the vector bundle E is now endowed with a Hermitian metric h.

The aim of this section is to prove Theorem B. The main ingredient required is Proposition 2.6 below, whose proof is based on an intrinsic expression of the Chern curvature of the universal line bundles $Q_{\rho,j} \to \mathbb{F}_\rho(E)$, which are equipped with the Hermitian metrics induced by h as described in Section 1.2.2.

2.2.1 Curvature intrinsic expression: universal line bundles

Here we rewrite Formula (1.13) more intrinsically. In order to do this, observe that Remark 1.10 shows that, for $1 \le j \le m$, if we define

$$\Xi^{\mathrm{vert}}_{\rho,j} := \Xi_{\rho,j} \circ (p_1 \otimes \overline{p_1})$$

and

$$\Xi^{\mathrm{hor}}_{\rho,j} := \Xi_{\rho,j} \circ (p_2 \otimes \overline{p_2}),$$

which are both sections of $\Lambda^{1,1} T^\vee_{\mathbb{F}_\rho(E)}$, we have

$$\Xi_{\rho,j} = \Xi^{\mathrm{vert}}_{\rho,j} + \Xi^{\mathrm{hor}}_{\rho,j}.$$

Moreover, again with the same choice of coordinates, pointwise at $(x_0, \mathbf{f}_0)$ we have

$$\left(\Xi^{\mathrm{hor}}_{\rho,j}\right)_{(x_0,\mathbf{f}_0)} = \frac{i}{2\pi} \sum_{\lambda=s_{j-1}+1}^{s_j} \Theta_{\lambda\lambda}(x_0), \tag{2.4}$$

this follows from Formula (1.13).

Next, for $x \in X$, let $\mathbf{f} \in \mathbb{F}_\rho(E_x)$ be given by a unitary basis $(v_1, \dots, v_r)$ of E_x. Define a section

$$\theta_j \colon \mathbb{F}_\rho(E) \to \Lambda^{1,1} T^\vee_{\mathbb{F}_\rho(E)}$$

as follows

$$\theta_j(x, \mathbf{f}) = \frac{i}{2\pi} \sum_{\lambda=s_{j-1}+1}^{s_j} \left\langle \pi^*\Theta(E,h)_{(x,\mathbf{f})} \cdot v_\lambda, v_\lambda \right\rangle_h.$$

Proposition 2.4. *The section θ_j is well defined, i.e. it does not depend upon the choice of a particular representative $\mathbf{v} = (v_1, \dots, v_r)$ for $\mathbf{f}$.*

Proof. Take a local normal frame $(e_1, \dots, e_r)$ of E at x such that $\mathbf{e} := (e_1(x), \dots, e_r(x))$ and $\mathbf{v}$ identify the same flag $\mathbf{f}$. For $\lambda = 1, \dots, r$, we have

$$v_\lambda = a_{1\lambda} e_1(x) + \cdots + a_{r\lambda} e_r(x) \tag{2.5}$$

where $A = (a_{pq})$ is the change of coordinates matrix. Since the unitary bases $\mathbf{e}$ and $\mathbf{v}$ give both the same flag, it follows that A is a block matrix with the following form

$$\begin{pmatrix} A_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_{mm} \end{pmatrix}, \tag{2.6}$$

where the diagonal block A_{jj} is again a unitary matrix of size $s_j - s_{j-1}$; in particular, for $1 \le \lambda \le r$ if the entry (λ, λ) hits the block A_{jj}, *i.e.* if $s_{j-1} < \lambda \le s_j$, Formula (2.5) reads as

$$v_\lambda = \sum_{\alpha = s_{j-1}+1}^{s_j} a_{\alpha\lambda} e_\alpha(x).$$

Using the local normal frame $\mathbf{e}$ (and omitting its dependence on x to simplify the notation), we get

$$\begin{aligned} &\frac{i}{2\pi} \sum_{\lambda = s_{j-1}+1}^{s_j} \left\langle \pi^* \Theta(E,h)_{(x,\mathbf{f})} \cdot v_\lambda, v_\lambda \right\rangle_h \\ &= \frac{i}{2\pi} \sum_{\lambda = s_{j-1}+1}^{s_j} \left\langle \pi^* \Theta(E,h)_{(x,\mathbf{f})} \cdot \left(\sum_{\alpha = s_{j-1}+1}^{s_j} a_{\alpha\lambda} e_\alpha \right), \sum_{\tilde\alpha = s_{j-1}+1}^{s_j} a_{\tilde\alpha\lambda} e_{\tilde\alpha} \right\rangle_h \\ &= \frac{i}{2\pi} \sum_{\alpha,\tilde\alpha = s_{j-1}+1}^{s_j} \underbrace{\sum_{\lambda = s_{j-1}+1}^{s_j} a_{\alpha\lambda} \overline{a_{\tilde\alpha\lambda}}}_{=\delta_{\alpha\tilde\alpha}} \left\langle \pi^* \Theta(E,h)_{(x,\mathbf{f})} \cdot e_\alpha, e_{\tilde\alpha} \right\rangle_h \\ &= \frac{i}{2\pi} \sum_{\alpha = s_{j-1}+1}^{s_j} \left\langle \pi^* \Theta(E,h)_{(x,\mathbf{f})} \cdot e_\alpha, e_\alpha \right\rangle_h, \end{aligned}$$

and the proposition follows. □

Of course, θ_j is smooth, hence it is a form in $\mathcal{A}^{1,1}(\mathbb{F}_\rho(E))$.

Lemma 2.5. *The equality*

$$\theta_j = \Xi^{\mathrm{hor}}_{\rho,j}$$

holds.

Proof. At any given $(x, \mathbf{f}) \in \mathbb{F}_\rho(E)$, choose $(e_1, \dots, e_r)$ to be a local normal frame for E at x such that $\mathbf{f}$ is given by $(e_1(x), \dots, e_r(x))$, and consider the induced holomorphic coordinates around $(x, \mathbf{f})$ as in Section 1.2.1.

Since the evaluation of θ_j in $(x, \mathbf{f})$ does not depend on the choice of the unitary basis defining $\mathbf{f}$, we have the following chain of equalities

$$\begin{aligned}\theta_j(x,\mathbf{f}) &= \frac{i}{2\pi}\sum_{\lambda=s_{j-1}+1}^{s_j}\langle \pi^*\Theta(E,h)_{(x,\mathbf{f})}\cdot e_\lambda(x), e_\lambda(x)\rangle_h \\ &= \frac{i}{2\pi}\sum_{\lambda=s_{j-1}+1}^{s_j}\pi^*\Theta_{\lambda\lambda}(x,\mathbf{f}) \\ &= \frac{i}{2\pi}\sum_{\lambda=s_{j-1}+1}^{s_j}\Theta_{\lambda\lambda}(x) \\ &= \left(\Xi^{\mathrm{hor}}_{\rho,j}\right)_{(x,\mathbf{f})},\end{aligned}$$

where the last equality follows from Formula (2.4). □

Now, in order to simplify the notation in what follows, let us relabel $\omega_j := \Xi^{\mathrm{vert}}_{\rho,j}$. The section ω_j is smooth since, for instance, by Lemma 2.5 it equals $\Xi_{\rho,j} - \theta_j$, which are two smooth $(1,1)$-forms on $\mathbb{F}_\rho(E)$.

Summing up, for $1 \le j \le m$, we have shown that

$$\Xi_{\rho,j} = \theta_j + \omega_j, \tag{2.7}$$

that is, the first Chern form of $(Q_{\rho,j}, h)$ can be written as a sum of a horizontal and a vertical part with respect to the natural splitting defined in Section 1.2.3, where the horizontal part contains the information coming from the curvature of (E,h) while the vertical part is the standard curvature of the determinant of the tautological successive quotients on a flag manifold.

2.2.2 Gysin formulæ for the curvature of universal line bundles

We are now in a good position to prove the following main technical proposition.

Proposition 2.6. *Let $F(\Xi_{\rho,1},\dots,\Xi_{\rho,m})$ be a complex homogeneous polynomial in the $(1,1)$-forms $\Xi_{\rho,1},\dots,\Xi_{\rho,m}$ on $\mathbb{F}_\rho(E)$. Then the push-forward*

$$(\pi_\rho)_* F(\Xi_{\rho,1},\dots,\Xi_{\rho,m})$$

is given by a universal (weighted) homogeneous polynomial formally evaluated in the Chern forms of (E,h).

In the statement, by *universal* we mean that this polynomial depends only on the shape of F and on the rank of E.

Proof. In order to simplify the notation, denote by π the projection π_ρ of the flag bundle $\mathbb{F}_\rho(E)$, and by $F(\Xi)$ the differential form $F(\Xi_{\rho,1},\dots,\Xi_{\rho,m})$. As usual, let d_ρ be the relative dimension of the flag bundle. Write $F(\Xi)$ as

$$\sum_{j_1+\dots+j_m=d_\rho+k} a_{(j_1,\dots,j_m)}\Xi^{j_1}_{\rho,1}\wedge\dots\wedge\Xi^{j_m}_{\rho,m},\quad a_{(j_1,\dots,j_m)}\in\mathbb{C},$$

where we can w.l.o.g. suppose that $0 \le k \le n$, otherwise the push-forward would be identically zero for obvious degree reasons. From now on, to simplify the notation, we omit the symbol $\wedge$ for the wedge product of forms and we use, where useful, the multi-index notation. Thanks to Formula (2.7), we can write

$$\begin{aligned} F(\Xi) &= \sum_{|J|=d_\rho+k} a_J(\theta_1+\omega_1)^{j_1}\dots(\theta_m+\omega_m)^{j_m} \\ &= \sum_{|J|=d_\rho+k} a_J \sum_{b_1=0}^{j_1}\binom{j_1}{b_1}\theta_1^{j_1-b_1}\omega_1^{b_1}\cdots\sum_{b_m=0}^{j_m}\binom{j_m}{b_m}\theta_m^{j_m-b_m}\omega_m^{b_m} \\ &= \sum_{|J|=d_\rho+k} a_J \sum_{b_1=0}^{j_1}\cdots\sum_{b_m=0}^{j_m}\binom{j_1}{b_1}\cdots\binom{j_m}{b_m}\theta_1^{j_1-b_1}\dots\theta_m^{j_m-b_m}\omega_1^{b_1}\dots\omega_m^{b_m}. \end{aligned}$$

Since π is a proper submersion, and by definition the push-forward $\pi_*F(\Xi)$ is given by integration along the fibers obtained by locally splitting the variables $(x,\mathbf{f})$, at $x\in X$, we have that

$$\pi_*F(\Xi)_x = \int_{\mathbf{f}\in\mathbb{F}_\rho(E_x)} F(\Xi)_{(x,\mathbf{f})},$$

where the second term stands, as already mentioned, for the integral of $F(\Xi)_{(x,\mathbf{f})}$ performed only on those differentials related to the variable $\mathbf{f}$. Therefore, for degree reasons, the only terms which can possibly survive after integration along the fibers are those for which $b_1+\dots+b_m=d_\rho$, since the ω_j's and only the ω_j's contain the relevant vertical differentials.

For the push-forward we thus obtain

$$\pi_*F(\Xi)_x = \sum_{\substack{|J|=d_\rho+k\\ b_1=0,\dots,j_1\\ \dots\\ b_m=0,\dots,j_m\\ b_1+\dots+b_m=d_\rho}} a_J\binom{j_1}{b_1}\cdots\binom{j_m}{b_m}\int_{\mathbb{F}_\rho(E_x)}\underbrace{\theta_1^{j_1-b_1}\dots\theta_m^{j_m-b_m}}_{=:\theta^{J-B}}\underbrace{\omega_1^{b_1}\dots\omega_m^{b_m}}_{=:\omega^B}.$$

What we want to do now is to explicitly write θ^{J-B} at an arbitrary point $(x,\mathbf{f})\in\mathbb{F}_\rho(E_x)$. Let $\mathbf{f}$ be given by a unitary basis $(v_1,\dots,v_r)$ of E_x, so that

$$\theta_\ell(x,\mathbf{f}) = \frac{i}{2\pi}\sum_{\lambda=s_{\ell-1}+1}^{s_\ell}\langle\pi^*\Theta(E,h)_{(x,\mathbf{f})}\cdot v_\lambda, v_\lambda\rangle_h.$$

Suppose as above we had fixed a local normal frame $(e_1,\dots,e_r)$ for E centered at $x\in X$, and let $v_\lambda=\sum_\nu v_\lambda^\nu\, e_\nu(x)$. Thus, we have

$$\theta_\ell(x,\mathbf{f}) = \frac{i}{2\pi}\sum_{\alpha,\beta=1}^{r}\left(\sum_{\lambda=s_{\ell-1}+1}^{s_\ell} v_\lambda^\alpha\bar v_\lambda^\beta\right)\Theta_{\beta\alpha}(x). \tag{2.8}$$

Then, we get the following expression for θ^{J-B}:

$$\begin{aligned} \theta^{J-B} &= \left(\frac{i}{2\pi}\right)^k\bigwedge_{\ell=1}^{m}\left(\sum_{\alpha_\ell,\beta_\ell=1}^{r}\left(\sum_{\lambda_\ell=s_{\ell-1}+1}^{s_\ell} v_{\lambda_\ell}^{\alpha_\ell}\bar v_{\lambda_\ell}^{\beta_\ell}\right)\Theta_{\beta_\ell\alpha_\ell}\right)^{j_\ell-b_\ell} \\ &= \left(\frac{i}{2\pi}\right)^k \sum_{\substack{\alpha_1^1,\beta_1^1,\dots,\alpha_1^{j_1-b_1},\beta_1^{j_1-b_1}=1,\dots,r\\ \dots\\ \alpha_m^1,\beta_m^1,\dots,\alpha_m^{j_m-b_m},\beta_m^{j_m-b_m}=1,\dots,r}} {}^B Q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}}\,\Theta_{\beta_1^1\alpha_1^1}\dots\Theta_{\beta_m^{j_m-b_m}\alpha_m^{j_m-b_m}}, \end{aligned}$$

where

$$ {}^B Q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}} = \prod_{\ell=1}^m \left[\left(\sum_{\lambda_\ell=s_{\ell-1}+1}^{s_\ell} v_{\lambda_\ell}^{\alpha_\ell^1} \bar v_{\lambda_\ell}^{\beta_\ell^1} \right) \cdots \left(\sum_{\lambda_\ell=s_{\ell-1}+1}^{s_\ell} v_{\lambda_\ell}^{\alpha_\ell^{j_\ell-b_\ell}} \bar v_{\lambda_\ell}^{\beta_\ell^{j_\ell-b_\ell}} \right) \right]. $$

Remark that the $\Theta_{\beta\alpha}$'s only depend on the point x, while the v_λ^ν's can be seen, by a slight abuse of notation, as variables of integration even if they have to be understood modulo the action (cf. with the construction of the matrix A in the proof of Proposition 2.4) of

$$ U(s_1-s_0)\times U(s_2-s_1)\times\cdots\times U(s_m-s_{m-1})\subset U(\underbrace{s_m-s_0}_{=r}) $$

which of course corresponds to the homogeneous presentation of the (possibly incomplete) flag manifold as $U(r)/U(s_1-s_0)\times\cdots\times U(s_m-s_{m-1})$.

At the end of the day, the integrals $\int_{\mathbb{F}_\rho(E_x)}\theta^{J-B}\omega^B$ are given by the following sum:

$$ \left(\frac{i}{2\pi}\right)^k \sum_{\substack{\alpha_1^1,\dots,\alpha_m^{j_m-b_m}=1,\dots,r\\ \beta_1^1,\dots,\beta_m^{j_m-b_m}=1,\dots,r}} {}^B q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}} \Theta_{\beta_1^1\alpha_1^1}\cdots\Theta_{\beta_m^{j_m-b_m}\alpha_m^{j_m-b_m}}, $$

where

$$ {}^B q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}} = \int_{\mathbb{F}_\rho(E_x)} {}^B Q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}}\,\omega^B. $$

The good news is that these coefficients ${}^B q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}}$ are universal, in the following sense: they do not depend anymore on the metric h, nor on the point $x\in X$, but only on the multi-index B and on the rank r of E.

Indeed, they might be calculated in the "absolute" case of the flag manifold $\mathbb{F}_\rho(\mathbb{C}^r)$, where $\mathbb{C}^r$ is endowed with the standard Euclidean metric, the v_λ^ν's are the element of a matrix in $U(r)$ representing the given flag, and where the top form ω^B is nothing else than the corresponding wedge product of the curvature forms of the determinant of the tautological quotients line bundles on $\mathbb{F}_\rho(\mathbb{C}^r)$ with respect to the natural metrics induced by the Euclidean metric of $\mathbb{C}^r$.

Summing up, we have shown that

$$ \pi_* F(\Xi)_x = \sum_{\substack{|J|=d_\rho+k\\ b_1=0,\dots,j_1\\ \cdots\\ b_m=0,\dots,j_m\\ b_1+\cdots+b_m=d_\rho}} \sum_{\substack{\alpha_1^1,\dots,\alpha_m^{j_m-b_m}=1,\dots,r\\ \beta_1^1,\dots,\beta_m^{j_m-b_m}=1,\dots,r}} a_J \binom{j_1}{b_1}\cdots\binom{j_m}{b_m} {}^B q^{\alpha_1^1\dots\alpha_m^{j_m-b_m}}_{\beta_1^1\dots\beta_m^{j_m-b_m}} \times\left(\frac{i}{2\pi}\Theta_{\beta_1^1\alpha_1^1}\right)\cdots\left(\frac{i}{2\pi}\Theta_{\beta_m^{j_m-b_m}\alpha_m^{j_m-b_m}}\right). $$

The above expression is thus given by evaluating a homogeneous polynomial $\tilde P$ of degree

$$ (j_1-b_1)+(j_2-b_2)+\cdots+(j_m-b_m)=|J|-|B|=k $$

on the entries $\Theta_{\beta\alpha}$'s of the matrix associated to the curvature $\Theta(E,h)_x$ with respect to the frame $(e_1(x),\dots,e_r(x))$.

This polynomial (*i.e.* its coefficients) is clearly independent of the point $x \in X$. Moreover, $\tilde{P}$ is of course also invariant under change of frame at x, because from its very definition in terms of push-forwards it is independent of the local frame chosen to make the computation. It follows, say from Chern–Weil theory (see for instance [GH78, p. 402]), that there exists a weighted homogeneous polynomial P such that globally on X

$$\pi_* F(\Xi) = P\big(c_1(E,h), \dots, c_r(E,h)\big), \tag{2.9}$$

where the $c_j(E,h)$'s are the Chern forms of (E,h). By construction, P is universal since it obviously depends only on F and the rank of E. □

Remark 2.7. Of course, since Chern forms may be expressed in terms of Segre forms, a completely analogous statement holds with a polynomial whose variables are now the Segre forms $s_j(E,h)$ of (E,h).

We shall see now how to compute the push-forward through the projection π_ρ of the differential form given by a complex homogeneous polynomial F in $\Xi_{\rho,1}, \dots, \Xi_{\rho,m}$.

Remark 2.8. Of course, for our purposes, it shall suffice to use some of the possible polynomials $\tilde{F}$ considered in Formula (2.1) only, namely those whose symmetries are of the form

$$\tilde{F}(\xi_1, \dots, \xi_r) = F\left(\dots, -\sum_{\ell = s_{j-1}+1}^{s_j} \xi_\ell, \dots\right) = F\big(c_1(Q_{\rho,1}), \dots, c_1(Q_{\rho,m})\big).$$

Observe however, that in the special case of complete flag bundles ($m = r$), the classes $\xi_1, \dots, \xi_r$ are not virtual and give actual cohomology classes. Indeed, for $1 \le j \le r$, we have the equality $\xi_j = -c_1(Q_j)$. Hence, the only relevant symmetry needed in this case is $\tilde{F} = (-1)^{d_\rho + k} F$.

The key result of this section is that, in fact, Darondeau–Pragacz formula (2.1) also holds pointwise at the level of differential forms, in the Hermitian setting, for polynomials $\tilde{F}$ satisfying the further symmetries considered in Remark 2.8.

Theorem 2.9. *Let (E,h) be a rank r Hermitian holomorphic vector bundle over a complex manifold X of dimension n, and let F be a complex homogeneous polynomial of degree $d_\rho + k$ in m variables. Then, we have the equality*

$$(\pi_\rho)_* F(\Xi_{\rho,1}, \dots, \Xi_{\rho,m}) = \Psi\big(s_1(E,h), \dots, s_n(E,h)\big).$$

Clearly, in the statement above, Ψ is the polynomial introduced in Remark 2.1 associated to the polynomial $\tilde{F}(\xi_\bullet) = F\big(c_1(Q_{\rho,\bullet})\big)$, as in Remark 2.8.

Proof. As in Proposition 2.6 denote by π the projection π_ρ to simplify the notation.

If $\eta \in \mathcal{A}^\bullet(X)$ is a closed form, then $[\eta]$ stands as usual for the cohomology class in $H^\bullet(X)$ represented by η. By Formula (2.1), it holds that

$$\begin{aligned}
[\pi_* F(\Xi_{\rho,1}, \dots, \Xi_{\rho,m})] &= \pi_*[F(\Xi_{\rho,1}, \dots, \Xi_{\rho,m})] \\
&= \pi_* \tilde{F}(\xi_1, \dots, \xi_r) \\
&= \Psi(s_1, \dots, s_n) \\
&= \big[\Psi\big(s_1(E,h), \dots, s_n(E,h)\big)\big] .
\end{aligned}$$

Hence, the difference

$$\pi_* F(\Xi_{\rho,1}, \dots, \Xi_{\rho,m}) - \Psi(s_1(E,h), \dots, s_n(E,h))$$

must be an exact global (k,k)-form on X. Recall that, by Proposition 2.6 and Formula (2.9), $\pi_* F(\Xi_{\rho,1}, \dots, \Xi_{\rho,m})$ is a universal weighted homogeneous polynomial $P = P(c_\bullet(E,h))$ of weighted degree $2k$ in the Chern forms of (E,h). If we express the Segre forms in terms of the Chern forms, the previous difference can be written as a complex weighted homogeneous polynomial

$$G(c_1(E,h), \dots, c_r(E,h)) = \sum_{k_1+2k_2+\dots+rk_r=k} g_{k_1\dots k_r} c_1(E,h)^{k_1} \wedge \dots \wedge c_r(E,h)^{k_r}$$

in the Chern forms. Note that G is universal (since P and Ψ are) in the sense that its coefficients $g_{k_1\dots k_r}$ do not depend upon E, nor X, but only upon r, n, and F. Recall that our aim is to show that G is in fact identically zero: following [Gul12], to achieve this we shall evaluate it on a particular vector bundle on a particular class of manifolds, as follows.

Take X to be any n-dimensional *projective* manifold and fix an ample line bundle A on X. Let ω_A be a metric on A with positive curvature. For $m_1, \dots, m_r$ positive integers, we define the totally split, rank r vector bundle

$$\mathcal{E} := A^{\otimes m_1} \oplus \dots \oplus A^{\otimes m_r},$$

and denote by $\omega_{\mathcal{E}}$ the natural induced metric on $\mathcal{E}$ by ω_A. Hence,

$$\begin{aligned} &G(c_1(\mathcal{E},\omega_{\mathcal{E}}), \dots, c_r(\mathcal{E},\omega_{\mathcal{E}})) \\ &= \sum_{k_1+2k_2+\dots+rk_r=k} g_{k_1\dots k_r} c_1(\mathcal{E},\omega_{\mathcal{E}})^{k_1} \wedge \dots \wedge c_r(\mathcal{E},\omega_{\mathcal{E}})^{k_r} \\ &= \sum_{k_1+2k_2+\dots+rk_r=k} g_{k_1\dots k_r} \bigwedge_{s=1}^{r} \left(\sum_{1\le j_1<\dots<j_s\le r} m_{j_1}\cdots m_{j_s} c_1(A,\omega_A)^s \right)^{k_s} \\ &= \left[\sum_{k_1+2k_2+\dots+rk_r=k} g_{k_1\dots k_r} \prod_{s=1}^{r} \left(\sum_{1\le j_1<\dots<j_s\le r} m_{j_1}\cdots m_{j_s} \right)^{k_s} \right] c_1(A,\omega_A)^k. \end{aligned}$$

Let $T_1, \dots, T_r$ be a set of formal variables, and consider the polynomial p defined by

$$p(T_1, \dots, T_r) = \sum_{k_1+2k_2+\dots+rk_r=k} g_{k_1\dots k_r} \prod_{s=1}^{r} \left(\sum_{1\le j_1<\dots<j_s\le r} T_{j_1} \dots T_{j_s} \right)^{k_s}. \tag{2.10}$$

We have by definition that

$$G(c_1(\mathcal{E},\omega_{\mathcal{E}}), \dots, c_r(\mathcal{E},\omega_{\mathcal{E}})) = p(m_1, \dots, m_r) c_1(A,\omega_A)^k$$

and, consequently, in cohomology it holds that

$$p(m_1, \dots, m_r) c_1(A)^k = [G(c_1(\mathcal{E},\omega_{\mathcal{E}}), \dots, c_r(\mathcal{E},\omega_{\mathcal{E}}))] = 0.$$

Since A is ample, the only possibility is that the polynomial p is zero for every choice of positive integers $m_1, \dots, m_r$. But the set of points in $\mathbb{C}^r$ whose coordinates are positive integers is Zariski dense, and thus p must be identically zero. Consequently, having the same coefficients as p, $G \equiv 0$ and this concludes the proof. □

Remark 2.10. As already said, Formula (2.1) is not the only possible instance of the Gysin formula given by Darondeau and Pragacz.

For example, we have already seen in Proposition 2.2 a universal Gysin formula (see Formula (2.3)) for flag bundles in terms of generalized Schur classes. Such formula is particularly useful since for instance it explicitly shows that one can obtain all the Schur polynomials in the Chern classes of E as push-forwards from the complete flag bundle (see Proposition 2.19 below for more details).

Clearly, our Theorem 2.9 proves the validity of Formula (2.3) at the level of differential forms for those $\tilde{F}$ as in Remark 2.8. We return on the extension of Formula (2.3) at the forms level in Section 2.4.

2.3 Push-forward formulæ for universal vector bundles

In this section, we prove Theorem A, which is an extension of the pointwise universal Gysin formula in Theorem 2.9 to polynomials in the Chern forms of all the universal vector bundles. The strategy of the proof is similar but, despite Theorem 2.9 where we use the curvature formulæ for the determinants of the successive quotients provided by [Dem88a], we need here curvature formulæ for all universal vector bundles. These formulæ are the core of this section, and we give them in the following.

2.3.1 Curvature of universal vector bundles

Given $\rho = (\rho_0, \dots, \rho_m)$ as in $(\star)$, consider the flag bundle $\pi_\rho \colon \mathbb{F}_\rho(E) \to X$. In the notation of Section 1.2, we now compute the Chern curvature tensors

$$\Theta(U_{\rho,l}, h_{\rho,l}) \quad \Theta(\pi_\rho^* E/U_{\rho,l}, H_{\rho,(m,l)}) \quad \Theta(U_{\rho,l}/U_{\rho,\ell}, H_{\rho,(l,\ell)})$$

of the universal vector bundles of $\mathbb{F}_\rho(E)$ endowed with the Hermitian metrics induced by h as in Remark 1.4.

Theorem 2.11. *Choose a point $(x_0, \mathbf{f}_0) \in \mathbb{F}_\rho(E)$ and let $(e_1, \dots, e_r)$ be a local normal frame of (E, h) at x_0 such that $\mathbf{f}_0 = [e_1(x_0), \dots, e_r(x_0)]_\rho$. Fix the local holomorphic coordinates (z, ζ) centered at $(x_0, \mathbf{f}_0)$ introduced in Section 1.2.1. Then:*

(i) for any $0 < l \le m$, the curvature $\Theta(U_{\rho,l}, h_{\rho,l})_{(x_0,\mathbf{f}_0)}$ equals

$$\sum_{r-\rho_l < \alpha,\beta \le r} \left(\Theta_{\beta\alpha} - \sum_{1 \le \lambda \le r-\rho_l} d\zeta_{\lambda\alpha} \wedge d\bar{\zeta}_{\lambda\beta} \right) \otimes \epsilon_\alpha^\vee \otimes \epsilon_\beta;$$

(ii) for any $0 < l < m$, the curvature $\Theta(\pi_\rho^* E/U_{\rho,l}, H_{\rho,(m,l)})_{(x_0,\mathbf{f}_0)}$ equals

$$\sum_{1 \le \alpha,\beta \le r-\rho_l} \left(\Theta_{\beta\alpha} + \sum_{r-\rho_l < \mu \le r} d\zeta_{\beta\mu} \wedge d\bar{\zeta}_{\alpha\mu} \right) \otimes \tilde{e}_\alpha^\vee \otimes \tilde{e}_\beta;$$

(iii) for any $0 < \ell < l < m$, the curvature $\Theta(U_{\rho,l}/U_{\rho,\ell}, H_{\rho,(l,\ell)})_{(x_0,\mathbf{f}_0)}$ equals

$$\sum_{r-\rho_l < \alpha,\beta \le r-\rho_\ell} \left(\Theta_{\beta\alpha} - \sum_{1 \le \lambda \le r-\rho_l} d\zeta_{\lambda\alpha} \wedge d\bar{\zeta}_{\lambda\beta} + \sum_{r-\rho_\ell < \mu \le r} d\zeta_{\beta\mu} \wedge d\bar{\zeta}_{\alpha\mu} \right) \otimes \tilde{\epsilon}_\alpha^\vee \otimes \tilde{\epsilon}_\beta.$$

By a slight abuse of notation, we have identified the $(1,1)$-form $\Theta_{\beta\alpha}$ with its pull-back $\pi_\rho^*\Theta_{\beta\alpha}$ along π_ρ.

Proof of Theorem 2.11 (i). Set $s := \rho_l$. By Remark 1.7 the curvature $\Theta(U_{\rho,l}, h_{\rho,l})$ is just the pull-back through the projection $\pi^\rho_{(0,s,r)}$ of the curvature $\Theta(\gamma_s, h_s)$ of the tautological vector bundle $(\gamma_s, h_s) \to \mathbb{G}_s(E)$. Thus, we can suppose that $\mathbb{F}_\rho(E) = \mathbb{G}_s(E)$ and that $U_{\rho,l} = \gamma_s$.

A local holomorphic frame for γ_s is given by the sections

$$\epsilon_\alpha(z,\zeta) = e_\alpha(z) + \sum_{1\le\lambda\le r-s} \zeta_{\lambda\alpha} e_\lambda(z), \quad \text{for } r-s<\alpha\le r.$$

Observe that, for the sequence $(0,s,r)$, Condition $(*)$ on λ and α for the coordinate $\zeta_{\lambda\alpha}$ becomes $1\le\lambda\le r-s<\alpha\le r$. We want to show that $(\epsilon_{r-s+1},\dots,\epsilon_r)$ is also a local normal frame of (γ_s,h_s) at $(x_0,\mathbf{f}_0)$.

To simplify the notation in what follows, we omit the dependence on the coordinates. The (α,β)-entry of the Hermitian matrix associated to h_s with respect to the coordinates $y := (z,\zeta)$ is

$$\begin{aligned}
&\langle \epsilon_\alpha, \epsilon_\beta\rangle \\
&= \left\langle e_\alpha + \sum_\lambda \zeta_{\lambda\alpha} e_\lambda, e_\beta + \sum_\mu \zeta_{\mu\beta} e_\mu \right\rangle \\
&= \langle e_\alpha, e_\beta\rangle + \sum_\mu \bar\zeta_{\mu\beta}\langle e_\alpha, e_\mu\rangle + \sum_\lambda \zeta_{\lambda\alpha}\langle e_\lambda, e_\beta\rangle + \sum_{\lambda,\mu}\zeta_{\lambda\alpha}\bar\zeta_{\mu\beta}\langle e_\lambda, e_\mu\rangle \\
&= \delta_{\alpha\beta} - \sum_{j,k} c_{jk\alpha\beta} z_j \bar z_k + \sum_\mu \bar\zeta_{\mu\beta}\delta_{\alpha\mu} + \sum_\lambda \zeta_{\lambda\alpha}\delta_{\lambda\beta} + \sum_{\lambda,\mu}\zeta_{\lambda\alpha}\bar\zeta_{\mu\beta}\delta_{\lambda\mu} + \mathrm{O}(|y|^3).
\end{aligned} \tag{2.11}$$

Now, observe that we can get rid of the terms $\sum \bar\zeta_{\mu\beta}\delta_{\alpha\mu}$ and $\sum \zeta_{\lambda\alpha}\delta_{\lambda\beta}$. Indeed, $\sum_\mu \bar\zeta_{\mu\beta}\delta_{\alpha\mu} = \bar\zeta_{\alpha\beta}$ but $\zeta_{\alpha\beta}=0$ since Condition $(*)$ on α and β contradicts the fact that the index α is greater than $r-s$. The same holds for β in the term $\sum_\lambda \zeta_{\lambda\alpha}\delta_{\lambda\beta}$. We have shown that

$$\langle \epsilon_\alpha, \epsilon_\beta\rangle = \delta_{\alpha\beta} - \sum_{1\le j,k\le n} c_{jk\alpha\beta} z_j \bar z_k + \sum_{1\le\lambda\le r-s} \zeta_{\lambda\alpha}\bar\zeta_{\lambda\beta} + \mathrm{O}(|y|^3)$$

thus, $(\epsilon_{r-s+1},\dots,\epsilon_r)$ is a normal coordinate frame at $(x_0,\mathbf{f}_0)$.

Therefore, the (β,α)-entry of the curvature matrix $\Theta(\gamma_s,h_s)_{(x_0,\mathbf{f}_0)}$ with respect to the coordinates y is

$$\sum_{1\le j,k\le n} c_{jk\alpha\beta}\, dz_j\wedge d\bar z_k - \sum_{1\le\lambda\le r-s} d\zeta_{\lambda\alpha}\wedge d\bar\zeta_{\lambda\beta}$$

and we have finished. □

Proof of Theorem 2.11 (ii). Set $s := \rho_l$ as before. By Remark 1.7 we can suppose that $(\pi_\rho^* E/U_{\rho,l}, H_{\rho,(m,l)})$ is the universal quotient (Q_s, H_s) over $\mathbb{G}_s(E)$.

The local holomorphic coordinates (z,ζ) on $\mathbb{G}_s(E)$ centered at $(x_0,\mathbf{f}_0)$ parameterize the s-planes $W\subset E_z$. Therefore, the local holomorphic frame (1.7) for Q_s is

$$\tilde e_\alpha(z,\zeta) = \text{image of } e_\alpha(z) \text{ in } E_z/W, \quad \text{for } 1\le\alpha\le r-s.$$

Let

$$0 \to \gamma_s \overset{\iota}{\hookrightarrow} \pi_\rho^* E \overset{p}{\twoheadrightarrow} Q_s \to 0$$

be the tautological exact sequence over $\mathbb{G}_s(E)$, and denote by $p^\star\colon Q_s \to \pi_\rho^* E$ the C^∞ orthogonal splitting of the natural projection p. We have

$$p^\star \cdot \tilde{e}_\alpha(z,\zeta) = e_\alpha(z) + \sum_{r-s<\mu\le r} u_{\alpha\mu}(z,\zeta)\epsilon_\mu(z,\zeta),$$

for some smooth functions $u_{\alpha\mu}$. To simplify the notation in what follows, we omit the dependence on the coordinates and we set $y := (z,\zeta)$. For $r-s<\beta$, the following chain of equalities hold

$$\begin{aligned}
0 &= \langle \tilde{e}_\alpha, p\cdot \epsilon_\beta\rangle \\
&= \langle p^\star \cdot \tilde{e}_\alpha, \epsilon_\beta\rangle \\
&= \left\langle e_\alpha + \sum_{r-s<\mu} u_{\alpha\mu}\epsilon_\mu, e_\beta + \sum_{\nu\le r-s} \zeta_{\nu\beta} e_\nu \right\rangle \\
&= \left\langle e_\alpha + \sum_{r-s<\mu} u_{\alpha\mu} e_\mu + \sum_{\substack{r-s<\mu\\ \sigma\le r-s}} u_{\alpha\mu}\zeta_{\sigma\mu} e_\sigma, e_\beta + \sum_{\nu\le r-s}\zeta_{\nu\beta} e_\nu \right\rangle \\
&= \langle e_\alpha, e_\beta\rangle + \sum_{\nu\le r-s} \bar{\zeta}_{\nu\beta}\langle e_\alpha, e_\nu\rangle + \sum_{r-s<\mu} u_{\alpha\mu}\langle e_\mu, e_\beta\rangle + \sum_{\substack{r-s<\mu\\ \nu\le r-s}} u_{\alpha\mu}\bar{\zeta}_{\nu\beta}\langle e_\mu, e_\nu\rangle \\
&\quad + \sum_{\substack{r-s<\mu\\ \sigma\le r-s}} u_{\alpha\mu}\zeta_{\sigma\mu}\langle e_\sigma, e_\beta\rangle + \sum_{\substack{r-s<\mu\\ \sigma\le r-s\\ \nu\le r-s}} u_{\alpha\mu}\zeta_{\sigma\mu}\bar{\zeta}_{\nu\beta}\langle e_\sigma, e_\nu\rangle \\
&= \delta_{\alpha\beta} - \sum_{j,k} c_{jk\alpha\beta} z_j \bar{z}_k + \bar{\zeta}_{\alpha\beta} + u_{\alpha\beta} \\
&\quad + \sum_{r-s<\mu} u_{\alpha\mu}\left(-\sum_{j,k} c_{jk\mu\beta} z_j\bar{z}_k + \sum_{\nu\le r-s} \zeta_{\nu\mu}\bar{\zeta}_{\nu\beta}\right) + \mathrm{O}(|y|^3).
\end{aligned}$$

Therefore, $u_{\alpha\beta} = -\bar{\zeta}_{\alpha\beta} + \mathrm{O}(|y|^2)$ and consequently,

$$p^\star \tilde{e}_\alpha = e_\alpha - \sum_{r-s<\mu} \left(\bar{\zeta}_{\alpha\mu} + \mathrm{O}(|y|^2)\right)\epsilon_\mu.$$

If $b \in \mathcal{A}^{1,0}\big(\mathbb{G}_s(E), \mathrm{Hom}(\gamma_s, Q_s)\big)$ denotes the second fundamental form of γ_s in $\pi_\rho^* E$ and $b^\star \in \mathcal{A}^{0,1}\big(\mathbb{G}_s(E), \mathrm{Hom}(Q_s,\gamma_s)\big)$ is the adjoint of b, then $-\iota\circ b^\star = \bar{\partial} p^\star$. Hence,

$$-\iota\circ b^\star \cdot \tilde{e}_\alpha|_{(x_0,\mathbf{f}_0)} = \bar{\partial} p^\star \tilde{e}_\alpha|_{(0,0)} = -\sum_{r-s<\mu} d\bar{\zeta}_{\alpha\mu}\otimes \epsilon_\mu.$$

We can write

$$b^\star_{(x_0,\mathbf{f}_0)} = \sum_{\substack{\alpha\le r-s\\ r-s<\mu}} d\bar{\zeta}_{\alpha\mu}\otimes \tilde{e}_\alpha^\vee \otimes \epsilon_\mu, \quad b_{(x_0,\mathbf{f}_0)} = \sum_{\substack{\beta\le r-s\\ r-s<\lambda}} d\zeta_{\beta\lambda}\otimes \epsilon_\lambda^\vee \otimes \tilde{e}_\beta.$$

Therefore, the curvature $\Theta(Q_s, H_s)_{(x_0,\mathbf{f}_0)}$ with respect to the coordinates y is

$$\left(\Theta(\pi_\rho^* E, \pi_\rho^* h)\big|_{Q_s} + b \wedge b^\star\right)_{(x_0,\mathbf{f}_0)}$$
$$= \sum_{\alpha,\beta} \left(\sum_{1\le j,k\le n} c_{jk\alpha\beta} dz_j \wedge d\bar{z}_k + \sum_{r-s<\mu\le r} d\zeta_{\beta\mu} \wedge d\bar{\zeta}_{\alpha\mu} \right) \otimes \tilde{e}_\alpha^\vee \otimes \tilde{e}_\beta$$

and we have done. □

Proof of Theorem 2.11 (iii). We are left only with the quotient of two proper tautological sub-bundles. Again by Remark 1.7 it is sufficient to compute the curvature of $(Q_{\mathbf{s}}, H_{\mathbf{s}})$ in a point of $\mathbb{F}_{\mathbf{s}}(E)$, where we have defined the sequence $\mathbf{s}$ to be $(0, s_1 = \rho_\ell, s_2 = \rho_l, r)$.

The local holomorphic coordinates (z, ζ) on $\mathbb{F}_{\mathbf{s}}(E)$ centered at $(x_0, \mathbf{f}_0)$ parameterize the flags $0 \subset V_{z,s_1} \subset V_{z,s_2} \subset E_z$. Thus, the local holomorphic frame (1.8) for $Q_{\mathbf{s}}$ is given by the sections

$$\tilde{\epsilon}_\alpha(z,\zeta) = \text{image of } \epsilon_\alpha(z,\zeta) \text{ in } V_{z,s_2}/V_{z,s_1}, \quad \text{for } r - s_2 < \alpha \le r - s_1.$$

Let

$$0 \to U_{\mathbf{s},1} \overset{\iota}{\hookrightarrow} U_{\mathbf{s},2} \overset{p}{\twoheadrightarrow} Q_{\mathbf{s}} \to 0$$

be a tautological exact sequence over $\mathbb{F}_{\mathbf{s}}(E)$, and let $p^\star \colon Q_{\mathbf{s}} \to U_{\mathbf{s},2}$ be the C^∞ orthogonal splitting of p. As before,

$$p^\star \cdot \tilde{\epsilon}_\alpha(z,\zeta) = \epsilon_\alpha(z,\zeta) + \sum_{r-s_1<\mu\le r} u_{\alpha\mu}(z,\zeta)\epsilon_\mu(z,\zeta),$$

for some smooth functions $u_{\alpha\mu}$. Set $y := (z, \zeta)$. For $r - s_1 < \beta \le r$, by using the computations made in (2.11), we have

$$\begin{aligned}
0 &= \langle \tilde{\epsilon}_\alpha, p \cdot \epsilon_\beta \rangle \\
&= \langle p^\star \cdot \tilde{\epsilon}_\alpha, \epsilon_\beta \rangle \\
&= \left\langle \epsilon_\alpha + \sum_{r-s_1<\mu} u_{\alpha\mu}\epsilon_\mu, \epsilon_\beta \right\rangle \\
&= \langle \epsilon_\alpha, \epsilon_\beta \rangle + \sum_{r-s_1<\mu} u_{\alpha\mu} \langle \epsilon_\mu, \epsilon_\beta \rangle \\
&= \delta_{\alpha\beta} - \sum_{j,k} c_{jk\alpha\beta} z_j \bar{z}_k + \bar{\zeta}_{\alpha\beta} + \sum_{\lambda \le r-s_2} \zeta_{\lambda\alpha}\bar{\zeta}_{\lambda\beta} \\
&\quad + u_{\alpha\beta} + \sum_{r-s_1<\mu} u_{\alpha\mu} \left(-\sum_{j,k} c_{jk\mu\beta} z_j \bar{z}_k + \sum_{\nu \le r-s_1} \zeta_{\nu\mu}\bar{\zeta}_{\nu\beta} \right) + \mathrm{O}(|y|^3).
\end{aligned}$$

Note that, unlike what happens in (2.11), the coordinate $\zeta_{\alpha\beta}$ is now well-defined. Indeed, the inequalities $\alpha \le r - s_1 < \beta$ are compatible with Condition $(*)$. However, the coordinate $\zeta_{\mu\beta}$ coming from the product $\langle \epsilon_\mu, \epsilon_\beta \rangle$ is zero. This because both μ and β are strictly greater than $r - s_1$. Thus, it can not happen that $\mu \le r - s_2 < \beta$, consequently Condition $(*)$ fails for $\zeta_{\mu\beta}$.

Therefore, $u_{\alpha\beta} = -\bar{\zeta}_{\alpha\beta} + \mathrm{O}(|y|^2)$ and

$$p^\star \tilde{\epsilon}_\alpha = \epsilon_\alpha - \sum_{r-s_1<\mu} \left(\bar{\zeta}_{\alpha\mu} + \mathrm{O}(|y|^2)\right) \epsilon_\mu.$$

Denoting by $b \in \mathcal{A}^{1,0}(\mathbb{F}_{\mathbf{s}}(E), \mathrm{Hom}(U_{\mathbf{s},1}, Q_{\mathbf{s}}))$ the second fundamental form of $U_{\mathbf{s},1}$ in $U_{\mathbf{s},2}$ and by $b^\star \in \mathcal{A}^{0,1}(\mathbb{F}_{\mathbf{s}}(E), \mathrm{Hom}(Q_{\mathbf{s}}, U_{\mathbf{s},1}))$ its adjoint, then

$$-\iota \circ b^\star \cdot \tilde{\epsilon}_\alpha|_{(x_0,\mathbf{f}_0)} = \bar{\partial} p^\star \tilde{\epsilon}_\alpha|_{(0,0)} = -\sum_{r-s_1<\mu} d\bar{\zeta}_{\alpha\mu} \otimes \epsilon_\mu.$$

Therefore, the curvature $\Theta(Q_{\mathbf{s}}, H_{\mathbf{s}})_{(x_0,\mathbf{f}_0)}$ with respect to the coordinates y is

$$\begin{aligned}&\left(\Theta(U_{\mathbf{s},2}, h_{\mathbf{s},2})\big|_{Q_{\mathbf{s}}} + b \wedge b^\star\right)_{(x_0,\mathbf{f}_0)} \\ &= \sum_{\alpha,\beta} \left(\Theta_{\beta\alpha} - \sum_{1\le\lambda\le r-s_2} d\zeta_{\lambda\alpha} \wedge d\bar{\zeta}_{\lambda\beta} + \sum_{r-s_1<\mu\le r} d\zeta_{\beta\mu} \wedge d\bar{\zeta}_{\alpha\mu}\right) \otimes \tilde{\epsilon}_\alpha^\vee \otimes \tilde{\epsilon}_\beta\end{aligned}$$

where the equality follows from the expressions

$$b^\star_{(x_0,\mathbf{f}_0)} = \sum_{\substack{r-s_2<\alpha\le r-s_1\\ r-s_1<\mu}} d\bar{\zeta}_{\alpha\mu} \otimes \tilde{\epsilon}_\alpha^\vee \otimes \epsilon_\mu, \quad b_{(x_0,\mathbf{f}_0)} = \sum_{\substack{r-s_2<\beta\le r-s_1\\ r-s_1<\lambda}} d\zeta_{\beta\lambda} \otimes \epsilon_\lambda^\vee \otimes \tilde{\epsilon}_\beta$$

and from the previous part (i). □

Parts (ii) and (iii) of the proof above follow the computations for the curvature of the tautological and universal quotient bundles of the Grassmann manifolds given in [Dem12, § 16.C].

Recall that, for $\rho = (0, r-1, r)$, [Mou04, Formula (2.1)] provides the curvature in a point of the universal quotient $T = \pi_\rho^* E^\vee / \mathcal{O}_{\mathbb{P}(E^\vee)}(-1)$ of the projectivized bundle $\mathbb{P}(E^\vee)$, computed with respect to the local frame given by the isomorphism

$$T \cong T_{\mathbb{P}(E^\vee)/X} \otimes \mathcal{O}_{\mathbb{P}(E^\vee)}(-1).$$

2.3.2 Curvature intrinsic expression: universal vector bundles

In the spirit of Section 2.2.1, we want to express intrinsically the curvature of the universal vector bundles. To do this we use the formulæ provided in our Theorem 2.11.

From now on, fix two indices ℓ and l such that $0 \le \ell < l \le m$. In order to simplify the notation, we denote by $\mathcal{E}$ the universal vector bundle $U_{\rho,l}/U_{\rho,\ell}$ of $\mathbb{F}_\rho(E)$. Finally, let H be the Hermitian metric naturally induced by h on $\mathcal{E}$, as mentioned in Remark 1.4.

As pointed out in Section 1.2.3, we have a natural (smooth) orthogonal splitting of the tangent bundle $T_{\mathbb{F}_\rho(E)}$ as the direct sum of the relative tangent bundle $T_{\mathbb{F}_\rho(E)/X}$ plus a horizontal part $T^{\perp_h}_{\mathbb{F}_\rho(E)/X}$, which depends only on h. Therefore, we can write the curvature tensor $\Theta(\mathcal{E}, H)$ as the sum of a "vertical" tensor

$$\Theta^{\mathrm{vert}}_{(\mathcal{E},H)} \in C^\infty\left(\mathbb{F}_\rho(E), \Lambda^{1,1} T^\vee_{\mathbb{F}_\rho(E)/X} \otimes \mathrm{End}(\mathcal{E})\right)$$

plus a "horizontal" tensor

$$\Theta^{\mathrm{hor}}_{(\mathcal{E},H)} \in C^\infty\left(\mathbb{F}_\rho(E), \Lambda^{1,1}(T^{\perp_h}_{\mathbb{F}_\rho(E)/X})^\vee \otimes \mathrm{End}(\mathcal{E})\right).$$

Our aim is to generalize what was done in Section 2.2.1 for the curvature of universal line bundles, giving an explicit expression of the tensor $\Theta^{\mathrm{hor}}_{(\mathcal{E},H)}$.

In the notation of Section 1.2.3, we have that

$$\Theta^{\mathrm{hor}}_{(\mathcal{E},H)} = \Theta(\mathcal{E},H) \circ (p_2 \otimes \bar{p}_2) \tag{2.12}$$

where the right hand side of Formula (2.12) is a slightly improper notation which means that we are composing the $(1,1)$-form part of $\Theta(\mathcal{E},H)$ with $p_2 \otimes \bar{p}_2$, where p_2 is the projection on $T^{\perp_h}_{\mathbb{F}_\rho(E)/X}$.

Moreover, fixed a point $(x_0, \mathbf{f}_0) \in \mathbb{F}_\rho(E)$ and with respect to the coordinates (z,ζ) introduced in Section 1.2.1, Theorem 2.11 and Formula (2.12) give

$$\Theta^{\mathrm{hor}}_{(\mathcal{E},H)}(x_0,\mathbf{f}_0) = \sum_{r-\rho_l<\alpha,\beta\le r-\rho_\ell} \Theta_{\beta\alpha} \otimes e^\vee_\alpha(x_0) \otimes e_\beta(x_0). \tag{2.13}$$

Observe that, whichever the local frame of $\mathcal{E}$, that is (1.6), (1.7) or (1.8), we can suppose that it coincides with $\big(e_{r-\rho_l+1}(x_0), \dots, e_{r-\rho_\ell}(x_0)\big)$ if evaluated in $(x_0,\mathbf{f}_0)$, since $(e_1,\dots,e_r)$ is a local normal frame at x_0.

Now, we define a section

$$\theta_{(\ell,l)} \colon \mathbb{F}_\rho(E) \to \Lambda^{1,1} T^\vee_{\mathbb{F}_\rho(E)} \otimes \mathrm{End}(\mathcal{E})$$

as follows. For $x \in X$, let $\mathbf{f} \in \mathbb{F}_\rho(E_x)$ be given by a unitary basis $(v_1,\dots,v_r)$ of E_x. We set

$$\theta_{(\ell,l)}(x,\mathbf{f}) = \frac{i}{2\pi} \sum_{r-\rho_l<\lambda,\mu\le r-\rho_\ell} \langle \pi_\rho^*\Theta(E,h)_{(x,\mathbf{f})} \cdot v_\lambda, v_\mu\rangle_h \otimes v^\vee_\lambda \otimes v_\mu. \tag{2.14}$$

Proposition 2.12. *The section $\theta_{(\ell,l)}$ is well defined, i.e. it does not depend upon the choice of a particular representative $\mathbf{v} = (v_1,\dots,v_r)$ for $\mathbf{f}$.*

Proof. Take a local normal frame $(e_1,\dots,e_r)$ of E at x such that $\mathbf{e} := (e_1(x),\dots,e_r(x))$ and $\mathbf{v}$ identify the same flag $\mathbf{f}$. As already observed in Proposition 2.4, we know that if the index λ is such that

$$s_{m-l} = r-\rho_l < \lambda \le r-\rho_\ell = s_{m-\ell}$$

then we can write

$$v_\lambda = \sum_{s_{m-l}<\alpha\le s_{m-\ell}} a_{\alpha\lambda} e_\alpha(x),$$

where the coefficients (a_{pq}) are the entries of the diagonal block matrix A as in (2.6).

Using the local normal frame $\mathbf{e}$ (and omitting its dependence on x to simplify the notation), we get

$$
\begin{aligned}
&\frac{i}{2\pi}\sum_{\lambda,\mu}\langle \pi_\rho^*\Theta(E,h)_{(x,\mathbf{f})}\cdot v_\lambda, v_\mu\rangle_h \otimes v_\lambda^\vee \otimes v_\mu \\
&= \frac{i}{2\pi}\sum_{\lambda,\mu}\left\langle \pi_\rho^*\Theta(E,h)_{(x,\mathbf{f})}\cdot\left(\sum_\alpha a_{\alpha\lambda}e_\alpha\right), \sum_\beta a_{\beta\mu}e_\beta\right\rangle_h \otimes \left(\sum_{\tilde\alpha} a_{\tilde\alpha\lambda}e_{\tilde\alpha}\right)^\vee \otimes \sum_{\tilde\beta} a_{\tilde\beta\mu}e_{\tilde\beta} \\
&= \frac{i}{2\pi}\sum_{\alpha,\tilde\alpha}\underbrace{\sum_\lambda a_{\alpha\lambda}\overline{a_{\tilde\alpha\lambda}}}_{=\delta_{\alpha\tilde\alpha}}\sum_{\tilde\beta,\beta}\underbrace{\sum_\mu a_{\tilde\beta\mu}\overline{a_{\beta\mu}}}_{=\delta_{\tilde\beta\beta}}\langle \pi_\rho^*\Theta(E,h)_{(x,\mathbf{f})}\cdot e_\alpha, e_\beta\rangle_h \otimes e_{\tilde\alpha}^\vee \otimes e_{\tilde\beta} \\
&= \frac{i}{2\pi}\sum_{\alpha,\beta}\langle \pi_\rho^*\Theta(E,h)_{(x,\mathbf{f})}\cdot e_\alpha, e_\beta\rangle_h \otimes e_\alpha^\vee \otimes e_\beta,
\end{aligned}
$$

where the indices of the summations are such that

$$\lambda,\mu,\alpha,\beta,\tilde\alpha,\tilde\beta = s_{m-l}+1,\dots,s_{m-\ell}.$$

Therefore, the proposition follows. □

Observe also that, by definition, the section $\theta_{(\ell,l)}$ is smooth, hence it belongs to the space $\mathcal{A}^{1,1}(\mathbb{F}_\rho(E), \mathrm{End}(\mathcal{E}))$.

Now we show that the section $\theta_{(\ell,l)}$ coincides in each point of the flag bundle with the h-horizontal part of the Chern curvature of $(\mathcal{E}, H)$.

Lemma 2.13. *The equality*

$$\theta_{(\ell,l)} = \frac{i}{2\pi}\Theta^{\mathrm{hor}}_{(\mathcal{E},H)}$$

holds.

Proof. At any given $(x,\mathbf{f}) \in \mathbb{F}_\rho(E)$, choose $(e_1,\dots,e_r)$ to be a local normal frame for E at x such that $\mathbf{f}$ is given by $(e_1(x),\dots,e_r(x))$, and consider the induced holomorphic coordinates around $(x,\mathbf{f})$ as in Section 1.2.1.

Since the evaluation of $\theta_{(\ell,l)}$ in $(x,\mathbf{f})$ does not depend on the choice of the unitary basis defining $\mathbf{f}$, we have the following chain of equalities

$$
\begin{aligned}
\theta_{(\ell,l)}(x,\mathbf{f}) &= \frac{i}{2\pi}\sum_{r-\rho_l<\lambda,\mu\le r-\rho_\ell}\langle \pi_\rho^*\Theta(E,h)_{(x,\mathbf{f})}\cdot e_\lambda(x), e_\mu(x)\rangle_h \otimes e_\lambda(x)^\vee \otimes e_\mu(x) \\
&= \frac{i}{2\pi}\sum_{r-\rho_l<\lambda,\mu\le r-\rho_\ell}\pi_\rho^*\Theta_{\mu\lambda}(x,\mathbf{f}) \otimes e_\lambda(x)^\vee \otimes e_\mu(x) \\
&= \frac{i}{2\pi}\sum_{r-\rho_l<\lambda,\mu\le r-\rho_\ell}\Theta_{\mu\lambda}(x) \otimes e_\lambda(x)^\vee \otimes e_\mu(x) \\
&= \frac{i}{2\pi}\Theta^{\mathrm{hor}}_{(\mathcal{E},H)}(x,\mathbf{f}),
\end{aligned}
$$

where the last equality follows from Formula (2.13). □

Similarly to what was done in Section 2.2.1, relabeling $\omega_{(\ell,l)} := \frac{i}{2\pi}\Theta^{\mathrm{vert}}_{(\mathcal{E},H)}$, we have shown that, for $0 \leq \ell < l \leq m$,

$$\frac{i}{2\pi}\Theta(\mathcal{E},H) = \theta_{(\ell,l)} + \omega_{(\ell,l)} \tag{2.15}$$

Observe that Formula (2.15) generalizes Formula (2.7).

2.3.3 Gysin formulæ for the curvature of universal vector bundles

Set $N := \binom{m+1}{2}$, and let $(\mathcal{E}_1, H_1), \ldots, (\mathcal{E}_N, H_N)$ be an enumeration of all the universal vector bundles, of ranks $r_1, \ldots, r_N$ respectively, over $\mathbb{F}_\rho(E)$. Let F be a complex weighted homogeneous polynomial of degree $2(d_\rho + k)$ in $r_1 + \cdots + r_N$ variables.

We state the following technical proposition which generalizes Proposition 2.6.

Proposition 2.14. *Let $F(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N))$ be a complex homogeneous polynomial in the Chern forms of the universal vector bundles on $\mathbb{F}_\rho(E)$. Then the push-forward*

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N))$$

is given by a universal (weighted) homogeneous polynomial formally evaluated in the Chern forms of (E, h).

Remark 2.15. In order to prove the proposition we follow the steps in the proof of Proposition 2.6, by using the results for the universal vector bundles obtained in Sections 2.3.1 and 2.3.2. However, unlike Proposition 2.6 where we give the explicit computations needed for the proof, here we provide general remarks on the key concepts that allow us to prove the theorem, since, in the end, the computations are similar and explicit formulæ are not strictly necessary.

Proof of Proposition 2.14. In the notation of Section 2.3.2, fix for a moment two indices $0 \leq \ell < l \leq m$ which identify a universal vector bundle $\mathcal{E}$ over $\mathbb{F}_\rho(E)$, equipped with the Hermitian metric H induced by h.

By Formula (2.15) we know that the curvature of $(\mathcal{E}, H)$ can be written as the sum of a horizontal tensor plus a vertical one (in the sense of Section 2.3.2). We denote them by θ and ω respectively, by ignoring the indices.

For $j = 1, \ldots, \mathrm{rk}\,\mathcal{E}$, recall the well-known equality from linear algebra

$$\begin{aligned} c_j(\mathcal{E},H) &= \mathrm{tr}_{\mathrm{End}(\Lambda^j\mathcal{E})}\left(\bigwedge^j \frac{i}{2\pi}\Theta(\mathcal{E},H)\right) \\ &= \frac{1}{j!}\det\begin{pmatrix} \mathrm{tr}(\theta+\omega) & j-1 & 0 & \cdots & \\ \mathrm{tr}(\theta+\omega)^2 & \mathrm{tr}(\theta+\omega) & j-2 & \cdots & \\ \vdots & \vdots & & \ddots & \vdots \\ \mathrm{tr}(\theta+\omega)^{j-1} & \mathrm{tr}(\theta+\omega)^{j-2} & & \cdots & 1 \\ \mathrm{tr}(\theta+\omega)^{j} & \mathrm{tr}(\theta+\omega)^{j-1} & & \cdots & \mathrm{tr}(\theta+\omega) \end{pmatrix} \end{aligned}$$

where we have used Formula (2.15).

Now, fix a point $x \in X$ and let $\mathbf{f}$ be any flag in the fiber $\mathbb{F}_\rho(E_x)$ given by a unitary basis $(v_1, \ldots, v_r)$ of E_x. Fixed a local normal frame $(e_1, \ldots, e_r)$ for E centered at $x \in X$, we write $v_\lambda = \sum_\nu v_\lambda^\nu e_\nu(x)$, where the coordinates v_λ^ν's are exactly as described in Proposition 2.6. Moreover, we can write θ (resp. ω) in matrix form with respect to the frame $(e_1(x), \ldots, e_r(x))$ as (θ_{pq}) (resp. ω_{uv}), where $p, q, u, v = 1, \ldots, \operatorname{rk} \mathcal{E}$.

Therefore, it is clear that each entry $\operatorname{tr}(\theta + \omega)^j$ is a polynomial in θ_{pq}'s and ω_{tu}'s, whose coefficients depend only on the rank of $\mathcal{E}$ and of the indices $0 \leq \ell < l \leq m$ defining the bundle $\mathcal{E}$. Consequently, by the chain of equalities above, $c_j(\mathcal{E}, H)$ is a polynomial of type $P^j(\theta_{pq}, \omega_{tu})$ whose coefficients are universal.

Now, Formula 2.14 gives us an explicit expression of θ for which, w.r.t. the frame $(e_1(x), \ldots, e_r(x))$, we get

$$\theta_{pq}(x, \mathbf{f}) = \frac{i}{2\pi} \sum_{\alpha,\beta=1}^{r} Q_{\alpha\beta}(v, \bar{v})\, \Theta_{\beta\alpha}(x), \tag{2.16}$$

where we have denoted by v the vector of coordinates v_λ^ν's. Observe that the polynomials $Q_{\alpha\beta}$'s have universal coefficients by Formula 2.14.

In order to compute the push-forward, by linearity we can of course suppose that the given polynomial F is just a monomial. Thus, it is sufficient to show that the push-forward through π_ρ of

$$\bigwedge_{s=1}^{N} \bigwedge_{j_s=1}^{r_s} c_{j_s}(\mathcal{E}_s, H_s)^{\wedge \ell_{j_s}} = \bigwedge_{s=1}^{N} \bigwedge_{j_s=1}^{r_s} P_s^{j_s}(\theta_{pq}^s, \omega_{tu}^s)^{\wedge \ell_{j_s}} \tag{2.17}$$

is a polynomial on the entries $\Theta_{\beta\alpha}$'s of the matrix associated to the curvature $\Theta(E, h)_x$ with respect to the frame $(e_1, \ldots, e_r)$.

We have observed in Formula (2.16) that θ_{pq}^s has universal polynomials in v and $\bar{v}$ as coefficients. Therefore, coupling this with the universality of the P^j's, after performing all the wedge powers and products in the right hand side of Formula (2.17), we can rewrite the latter as a polynomial A in the $\Theta_{\beta\alpha}$'s and ω_{tu}^s's. Moreover, since all the polynomials involved in the right hand side of Formula (2.17) are universal, and we are performing wedge powers and products of them, we deduce that the coefficients of A are universal polynomials in v and $\bar{v}$.

As already pointed out in Proposition 2.6, recall that the $\Theta_{\beta\alpha}$'s only depend on the point x, while the v_λ^ν's can be seen as variables of integration. Moreover, only the ω_{tu}^s's contain the vertical differentials which can be integrated along the fibers of the submersion π_ρ.

By applying $(\pi_\rho)_*$ to Formula (2.17), we have expressed

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N))$$

as a polynomial in the $\Theta_{\beta\alpha}$'s, whose coefficients are integrals over $\mathbb{F}_\rho(E_x)$ of universal polynomials in the variables v_λ^ν's, and the volume forms are products of the ω_{tu}^s. By the same reasons as in Proposition 2.6, these integrals can be computed over the flag manifold $\mathbb{F}_\rho(\mathbb{C}^r)$, by applying a unitary transformation from (E_x, h_x) onto $\mathbb{C}^r$ equipped with the standard metric. This concludes the proof. □

Now we see how to explicitly compute the push-forward

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N)) \in \mathcal{A}^{k,k}(X).$$

At the cohomology level, we formally evaluate F in the Chern classes of $\mathcal{E}_1, \ldots, \mathcal{E}_N$. This gives a cohomology class $F\big(c_\bullet(\mathcal{E}_1), \ldots, c_\bullet(\mathcal{E}_N)\big)$ in the group $H^{2(d_\rho+k)}\big(\mathbb{F}_\rho(E)\big)$, which we write as a polynomial $\tilde{F}$ in terms of the Chern roots $\xi_1, \ldots, \xi_r$ of $\pi_\rho^* E^\vee$. Thus, we have an equality

$$F\big(c_\bullet(\mathcal{E}_1), \ldots, c_\bullet(\mathcal{E}_N)\big) = \tilde{F}(\xi_1, \ldots, \xi_r).$$

Darondeau–Pragacz formula (2.1) gives us a universal weighted homogeneous polynomial Ψ of degree $2k$ such that

$$(\pi_\rho)_* \tilde{F}(\xi_1, \ldots, \xi_r) = \Psi\big(s_1(E), \ldots, s_n(E)\big).$$

By expressing the Segre classes in terms of the Chern classes, we get a polynomial Φ such that $\Phi\big(c_\bullet(E)\big) = \Psi\big(s_\bullet(E)\big)$. We thus have the equation

$$(\pi_\rho)_* F\big(c_\bullet(\mathcal{E}_1), \ldots, c_\bullet(\mathcal{E}_N)\big) = \Phi\big(c_1(E), \ldots, c_r(E)\big). \tag{2.18}$$

Observe that the equation above can be recovered also by using Formula 2.3.

The next result, which is the main theorem of this section, completely translates Formula (2.18) at the level of differential forms and generalizes Theorem 2.9.

Theorem 2.16. *Let (E, h) be a rank r Hermitian holomorphic vector bundle over a complex manifold X of dimension n, and let F be a complex homogeneous polynomial of degree $d_\rho + k$ in $r_1 + \cdots + r_N$ variables. Then, we have the equality*

$$(\pi_\rho)_* F\big(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N)\big) = \Phi\big(c_1(E, h), \ldots, c_r(E, h)\big).$$

Proof. The difference

$$(\pi_\rho)_* F\big(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N)\big) - \Phi\big(c_\bullet(E, h)\big)$$

is of course an exact global (k, k)-form on X. By Proposition 2.14,

$$(\pi_\rho)_* F\big(c_\bullet(\mathcal{E}_1, H_1), \ldots, c_\bullet(\mathcal{E}_N, H_N)\big)$$

is a universal homogeneous polynomial of weighted degree $2k$ in the Chern forms of (E, h). Hence, the previous difference can be written as a complex weighted homogeneous polynomial G in the Chern forms of (E, h), whose coefficients depend only upon r, n and F.

From now on, the proof is exactly the same as the proof of Theorem 2.9. Indeed, to show that $G\big(c_\bullet(E, h)\big)$ is identically zero, we just need that it is exact and that G is a polynomial whose coefficients are universal. □

Explicit computations which use Theorem 2.16 will be given in Section 3.2.1.

Remark 2.17. Theorem 2.9 is a particular case of Theorem 2.16. Indeed, Theorem 2.9 considers only polynomials in the first Chern forms of the successive quotients

$$\pi_\rho^* E \Big/ U_{\rho, m-1}, \ldots, U_{\rho,2} \Big/ U_{\rho,1}, U_{\rho,1}$$

while Theorem 2.16 allows to compute the push-forward of any polynomial in the Chern forms of all the possible universal vector bundles.

2.4 Alternative expression of the push-forward formulæ

In the notation of Proposition 2.2, we now see how to give an analogue of Formula 2.3 to the differential forms level. As always, suppose that $(E, h) \to X$ is a Hermitian holomorphic vector bundle of rank r.

Given a homogeneous polynomial F as in Theorem 2.16, we want to give an explicit expression of the push-forward

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \dots, c_\bullet(\mathcal{E}_N, H_N))$$

in terms of generalized Schur forms of (E, h).

Clearly, at the cohomology level we have that

$$F(c_\bullet(\mathcal{E}_1), \dots, c_\bullet(\mathcal{E}_N)) \in H^{2(d_\rho + k)}(\mathbb{F}_\rho(E)).$$

Therefore, there exists a polynomial in the virtual Chern roots of $\pi_\rho^* E^\vee$ called

$$\tilde{F}(\xi_1, \dots, \xi_r) = \sum_{|\lambda| = d_\rho + k} b_\lambda \xi_1^{\lambda_1} \cdots \xi_r^{\lambda_r},$$

which has the appropriate symmetries for which

$$\tilde{F}(\xi_1, \dots, \xi_r) = F(c_\bullet(\mathcal{E}_1), \dots, c_\bullet(\mathcal{E}_N)),$$

i.e., the polynomial $\tilde{F}(\xi_1, \dots, \xi_r)$ can be considered a cohomology class of $\mathbb{F}_\rho(E)$.

In particular, if $m = r$ the Chern roots of $\pi^* E^\vee$ are not virtual, cf. with Remark 2.8.

Since the explicit expression of the push-forward in cohomology is obviously independent of the method we use to compute it, from Formula 2.3 and Theorem 2.16 it follows the identity

$$(\pi_\rho)_* F(c_\bullet(\mathcal{E}_1, H_1), \dots, c_\bullet(\mathcal{E}_N, H_N)) = \sum_{|\lambda| = d_\rho + k} b_\lambda s_{(\lambda - \nu)^\leftarrow}(E, h), \tag{2.19}$$

where ν is given as in Proposition 2.2 and the b_λ's are the coefficients of $\tilde{F}$. Obviously, the right hand side of Formula (2.19) is the form $\Phi(c_\bullet(E, h))$ given by Theorem 2.16.

In the next result, by means of Formula (2.19) we prove that each Schur form can be obtained as a push-forward from some flag bundle.

Let σ be a partition in $\Lambda(k, r)$. By the Jacobi–Trudi identities we know that

$$S_\sigma(E, h) = (-1)^{|\sigma|} s_{\sigma'}(E, h) \tag{2.20}$$

where σ' is the conjugate partition of σ, cf. with Example 1.13.

Remark 2.18. Observe that we can canonically associate to the conjugate partition $\sigma' \in \Lambda(k, r) \subset \mathbb{N}^k$ an element $\tilde{\sigma} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_r)$ of $\mathbb{N}^r$ as follows. If

- $k < r$, then we set $\tilde{\sigma} = (\sigma'_1, \dots, \sigma'_k, \sigma'_{k+1} = 0, \dots, \sigma'_r = 0)$;
- $k = r$, then $\tilde{\sigma}$ is σ';

- $k > r$, then, since $r \geq \sigma_1 \geq \cdots \geq \sigma_k \geq 0$, by definition σ' must be of the form $(\sigma'_1, \ldots, \sigma'_r, 0, \ldots, 0)$. Therefore we set $\tilde{\sigma} = (\sigma'_1, \ldots, \sigma'_r)$.

In all of the three cases above, it follows from the Definition 1.12 that $s_{\tilde{\sigma}}(E, h) = s_{\sigma'}(E, h)$.

The next proposition shows that the Schur form $S_\sigma(E, h)$ can be obtained as a push-forward from the complete flag bundle associated to E. We follow the notation introduced in Remark 2.18.

Proposition 2.19. *Let σ be a partition in $\Lambda(k, r)$ and let $\lambda \in \mathbb{Z}^r$ be the sequence whose j-th element is $\lambda_j = \tilde{\sigma}_{r-j+1} + j - 1$. Then*

$$\pi_*\big[(-1)^{|\lambda|+|\sigma|}\Xi_1^{\wedge\tilde{\sigma}_r} \wedge \cdots \wedge \Xi_r^{\wedge(\tilde{\sigma}_1+r-1)}\big] = S_\sigma(E, h) \tag{2.21}$$

where π is the natural projection of the complete flag bundle $\mathbb{F}(E)$, and the Ξ_j's are the first Chern forms of its tautological successive quotients Q_j's.

Proof. The sequence λ is defined in such a way that $\tilde{\sigma} = (\lambda - \nu)^{\leftarrow}$, where $\nu = (0, 1, \ldots, r-1)$ is as in Formula (2.2). Therefore, by Formula (2.19) it follows that

$$\pi_*\big[(-\Xi_1)^{\lambda_1} \wedge \cdots \wedge (-\Xi_r)^{\lambda_r}\big] = s_{(\lambda-\nu)^{\leftarrow}}(E, h) = (-1)^{|\sigma|} S_\sigma(E, h)$$

where the last equality follows from Formula (2.20) and Remark 2.18. □

Observe that Proposition 2.19 is an alternative expression of the Jacobi–Trudi identity for differential forms given in [Fin21, Theorem 3.18]. Therefore, these two results are both a special case of Theorem 2.16.

Chapter 3

Positivity of characteristic forms for positive vector bundles

In this chapter, by means of the universal Gysin formulæ obtained in Chapter 2, we show the (strong) positivity of several characteristic differential forms built from the Chern curvature of Griffiths semipositive vector bundles. This gives a partial (but in a sense stronger) confirmation to a conjecture proposed by Griffiths in the late sixties, which has raised interest in the past (see, for instance, [Gri69, FL83, DPS94]) as well as in recent years (see, for instance, [Gul12, Fin21, Li21]).

In Section 3.1 we describe the above mentioned Griffiths' conjecture ([Gri69]), which concerns the positivity of the characteristic forms coming from positive vector bundles. In Section 3.2 we prove Theorem C and in Section 3.3 we prove Theorem D. These two results provide characteristic forms whose positivity was not previously known. Finally, Section 3.4 collects several results on Griffiths' conjecture in other literature, and some concluding remarks and open questions which may serve for further developments of this topic.

3.1 About Griffiths' conjecture on positive polynomials

As we saw in the introduction, Griffiths conjectured (and proved partially) in [Gri69] that given any rank r Hermitian holomorphic positive vector bundle on a projective manifold, the polynomials belonging to $\Pi(r)$ whenever evaluated on its Chern classes have to return a positive number once integrated over any subvariety of the correct dimension. A full proof of this conjecture is given in [FL83] in the more general setting of ample vector bundles (see also [DPS94] for the even more general context of E nef and X compact Kähler).

Actually, in [FL83] a more universal problem is considered and settled, *i.e.* to characterize precisely the *numerically positive polynomials for ample vector bundles of rank* r. These are defined to be those weighted homogeneous polynomials say of degree $2n$ that whenever evaluated on the Chern classes of any rank r ample vector bundle $\mathcal{V}$ over an irreducible projective variety of dimension n give a positive number. Once again, the characterization is that numerically positive polynomials for ample vector bundles of rank r are exactly the non zero positive polynomials.

Remark 3.1. It is observed in [FL83, Remark (1)] that if a weighted homogeneous polynomial P of degree $2n$ is not positive, *i.e.* it does not belong to $\Pi(r)$, then there exists a *smooth* projective manifold of dimension n, and an ample vector bundle of rank r over it, such that when one evaluates this polynomial in its Chern classes and integrates over the manifold, one gets a negative number. Moreover, such vector bundle is constructed as a quotient of a direct sum of very ample line bundles.

The upshot is that if we want to show that a weighted homogeneous polynomial is positive it suffices to show that it is a numerically positive polynomial for ample vector bundles over *smooth* projective manifolds. This will be useful later during the proof of Theorem 3.6.

Now, let $E \to X$ be a holomorphic vector bundle endowed with a Griffiths (semi)positive Hermitian metric h.

It is then natural to ask whether in this Hermitian setting the Fulton–Lazarsfeld–Demailly–Peternell–Schneider theorem holds pointwise for Chern forms, and this is also a question raised by Griffiths in the same paper.

Recall that in Definition 1.15 we have given the three main notions of positivity for differential (k,k)-forms.

Question 1 ([Gri69])**.** Given a Griffiths (semi)positive Hermitian holomorphic vector bundle (E,h), is it true that the positive polynomials evaluated on the Chern forms of E give rise to positive forms?

Remark 3.2. Coming back to Remark 3.1, we see that given P a weighted homogeneous polynomial of degree $2n$ which is not positive, there exists a rank r holomorphic Hermitian vector bundle (E,h) over a smooth projective manifold X of dimension n whose Chern curvature is Griffiths (as well as dual Nakano) positive and such that the corresponding characteristic form obtained by computing P in the Chern forms of (E,h) is not a positive (volume) form.

This is because one can endow E with the quotient metric of a positively curved direct sum metric on the direct sum of the very ample line bundles in question. Such a metric, begin a quotient of a positively curved (in any sense) metric, is both Griffiths and dual Nakano positive (but not Nakano positive, in general). The corresponding volume form $P(c_\bullet(E,h))$ has negative total mass, and hence must be negative somewhere.

This means that, even in the pointwise Hermitianized case considered by Griffiths, the cone of positive polynomials is the largest possible for which one can hope such a result.

Griffiths, in *loc. cit.*, answered in the affirmative to this question in the special case of the second Chern form of a rank 2 Griffiths positive holomorphic vector bundle (for the first Chern form the answer is trivially yes). Remark that this question gives, under the stronger hypothesis of Griffiths positivity, a stronger answer than its cohomological version stated earlier, since —as observed— a positive polynomial in the Chern form is a special representative in cohomology of the corresponding positive polynomial in the Chern classes.

In the last recent years there have been several partial results towards a fully affirmative answer to Griffiths' question. First, [Gul12] (see also [Div16] for a more direct proof of the main technical result needed, as well as [Mou04] for similar, and somehow more general, computations) proved that the answer is affirmative in the special case of singed Segre classes.

Then, [Li21] proved the full statement but under the stronger assumption of Bott–Chern nonnegativity for (E, h) (see Section 3.4.1). We refer to *ibid.* for the definition of this variant of Hermitian positivity, which has been observed to be indeed equivalent to dual Nakano semipositivity in [Fin21].

Other related interesting results about finding positive representatives (not necessarily coming from the given positively curved metric) of the Schur polynomials in the Chern classes are obtained in [Pin18, Xia22].

For an even more recent result in the case of (dual) Nakano positive vector bundles, see [Fin21].

3.2 A family of strongly positive characteristic forms

Here, we are concerned with the original Question 1, which is still very much open: one can construct indeed (local, say over a ball) examples of Hermitian holomorphic vector bundles which are Griffiths positive but not Nakano nor dual Nakano positive, see for instance [Fin21, Proposition 2.9]. Our Theorem 2.9 allows us indeed to confirm the strong positivity of quite a few new positive combinations of Schur polynomials in the Chern forms, as follows.

Let (E, h) be a Griffiths semipositive vector bundle of rank r over a complex manifold X of dimension n. Consider the flag bundle $\pi_\rho \colon \mathbb{F}_\rho(E) \to X$ and, for $\mathbf{a} \in \mathbb{N}^r$ satisfying Condition (1.9), let $Q_\rho^{\mathbf{a}} \to \mathbb{F}_\rho(E)$ be the line bundle introduced in Section 1.2.2.

The first observation is contained in the following.

Proposition 3.3 ([Dem88a, Lemma 3.7 (a), Formula (4.9)]). *If*

$$\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$$

is non increasing, then $Q_\rho^{\mathbf{a}} \to \mathbb{F}_\rho(E)$ endowed with the natural Hermitian metric induced by h is a semipositive line bundle.

In the notation of Section 1.2.2, this means precisely that the Chern curvature $\Xi_\rho^{\mathbf{a}} = c_1(Q_\rho^{\mathbf{a}}, h)$ is a closed positive $(1,1)$-form.

Let us recall (see Remark 1.19) that strongly positive forms are positive, but the converse is not true in general. However, strongly positive (k,k)-forms and positive (k,k)-forms do coincide for $k = 0, 1, n-1, n$. Thus, $\Xi_\rho^{\mathbf{a}}$ is also a strongly positive $(1,1)$-form.

Since the wedge product of strongly positive forms is again strongly positive (see Definition 1.15), then all wedge powers of $\Xi_\rho^{\mathbf{a}}$ are again strongly positive.

Now, by Proposition 1.25 we have that the push-forward of a closed strongly positive form under a proper holomorphic submersion is again a closed strongly positive form. Thus, we obtain immediately the next proposition.

Proposition 3.4. *If $(E, h) \to X$ is a Griffiths semipositive vector bundle, then the closed forms*

$$(\pi_\rho)_*(\Xi_\rho^{\mathbf{a}})^{\wedge(d_\rho + k)}, \quad a_1 \geq a_2 \geq \dots \geq a_r \geq 0,$$

where d_ρ is the relative dimension and k is a non negative integer, are closed strongly positive (k,k)-forms.

Remark 3.5. If it happens that the chain of inequalities in the above statement is not strictly decreasing where prescribed by Condition (1.12), then the push-forward is identically zero. This is because in this case the curvature $\Xi_\rho^{\mathbf{a}}$ has some vertical zero eigenvalue in each fiber, thanks to Formula (1.11). Therefore, the vertical top form against which we integrate to obtain the push-forward is identically zero being, modulo a factor, the determinant of the vertical part of the curvature. So, in what follows, we can consider without loss of generality only weights $\mathbf{a} \in \mathbb{N}^r$ satisfying Condition (1.9) and such that $a_{s_1} > a_{s_2} > \cdots > a_{s_m}$.

Recall that we have denoted by $\xi_1, \ldots, \xi_r$ the Chern roots of $\pi_\rho^* E^\vee$. Now, we come to the main result of this section.

Theorem 3.6. *Let (E,h) be a Griffiths semipositive Hermitian holomorphic vector bundle of rank r over a complex manifold X of dimension n.*

For every $\mathbf{a} \in \mathbb{N}^r$ satisfying Conditions (1.9) and (1.12), and for $0 \leq k \leq n$, the differential form

$$(\pi_\rho)_*(\Xi_\rho^{\mathbf{a}})^{\wedge(d_\rho+k)}$$

is a closed strongly positive (k,k)-form on X belonging to the positive convex cone $\Pi(E,h)$ spanned by the Schur forms of (E,h).

Moreover, given the polynomial

$$P(t_1, \ldots, t_r) = \left(-\sum_{j=1}^{m} \sum_{\lambda=s_{j-1}+1}^{s_j} a_{s_j} t_\lambda \right)^{d_\rho+k},$$

the explicit expression of the push-forward can be obtained either:

- *by formally evaluating in the Segre forms of (E,h) the right hand side of Formula (2.1) with $\tilde{F} = P$;*
- *by formally evaluating in the Segre forms of (E,h) the right hand side of Formula (2.3), which is applied to $\tilde{F}(\xi_\bullet) = P(\xi_\bullet)$.*

This theorem covers in particular Guler's work [Gul12], which concerned push-forwards from the projectivized bundle.

Remark 3.7. Observe that it is in some sense more natural to obtain that these forms are strongly positive rather than merely positive. This is because, as said earlier, positive polynomials are stable under products and so do strongly positive forms, while a product of positive forms is not necessarily still positive.

Proof. By definition, since $\Xi_\rho^{\mathbf{a}} = c_1(Q_\rho^{\mathbf{a}}, h)$ we have that

$$(\Xi_\rho^{\mathbf{a}})^{\wedge(d_\rho+k)} = (a_{s_1}\Xi_{\rho,1} + \cdots + a_{s_m}\Xi_{\rho,m})^{\wedge(d_\rho+k)},$$

and thanks to Theorem 2.9 and Formula (2.19) applied to

$$F(\Xi_{\rho,1}, \ldots, \Xi_{\rho,m}) = (a_{s_1}\Xi_{\rho,1} + \cdots + a_{s_m}\Xi_{\rho,m})^{\wedge(d_\rho+k)}$$

we get, respectively, the two explicit expressions claimed at the end of the statement.

The closedness and strong positivity of $\Phi^k_{\mathbf{a}}(E,h)$ are the content of Proposition 3.4.

By Remark 2.7, there exists a unique homogeneous polynomial $\Psi^k_{\mathbf{a}}$ of weighted degree $2k$ such that

$$(\pi_\rho)_*(\Xi^{\mathbf{a}}_\rho)^{\wedge(d_\rho+k)} = \Psi^k_{\mathbf{a}}\big(s_1(E,h),\dots,s_n(E,h)\big).$$

We now want to show that $(\pi_\rho)_*(\Xi^{\mathbf{a}}_\rho)^{\wedge(d_\rho+k)}$ can be written as a positive linear combination of Schur forms. To do this, let $\Phi^k_{\mathbf{a}}$ be the *unique* polynomial such that

$$\Psi^k_{\mathbf{a}}\big(s_1(E,h),\dots,s_n(E,h)\big) = \Phi^k_{\mathbf{a}}\big(c_1(E,h),\dots,c_r(E,h)\big).$$

Observe that obviously the polynomial $\Phi^k_{\mathbf{a}}$ does not depend on the particular vector bundle considered, nor on the particular given base manifold, as usual. What we want to show will then follow from the Fulton–Lazarsfeld theorem [FL83] if we can prove that $\Phi^k_{\mathbf{a}}$ is a numerically positive polynomial for ample vector bundles of rank r over smooth projective manifolds, see Remark 3.1.

So, take any rank r ample vector bundle $\mathcal{V}$ over a k-dimensional projective manifold Z. By [Dem88b, Lemma 4.1], the corresponding line bundle $Q^{\mathbf{a}}_\rho$ over $\mathbb{F}_\rho(\mathcal{V})$ is ample, and it can be therefore endowed with a smooth Hermitian metric $h_{\mathcal{V},\mathbf{a}}$ whose Chern curvature $i\Theta(Q^{\mathbf{a}}_\rho, h_{\mathcal{V},\mathbf{a}})$ is *strictly* positive, *i.e.* a Kähler form. But then, $(\pi_\rho)_*c_1(Q^{\mathbf{a}}_\rho, h_{\mathcal{V},\mathbf{a}})^{\wedge(d_\rho+k)}$, is a closed positive *nowhere zero* (k,k)-form representing the cohomology class $\Phi^k_{\mathbf{a}}(c_1(\mathcal{V}),\dots,c_r(\mathcal{V}))$. In particular, being represented by a non zero positive (k,k)-form, we have that

$$\int_Z \Phi^k_{\mathbf{a}}\big(c_1(\mathcal{V}),\dots,c_r(\mathcal{V})\big) > 0,$$

as desired. □

As a byproduct of the proof above one immediately obtains the following statement for ample vector bundles in the same spirit of [Pin18, Xia22].

Theorem 3.8. *Let $E \to X$ be an ample vector bundle of rank r over a projective manifold. For every $\mathbf{a} \in \mathbb{N}^r$ satisfying Conditions* (1.9) *and* (1.12)*, and for every $k = 0,\dots,n = \dim X$, the (k,k)-cohomology classes $(\pi_\rho)_*c_1(Q^{\mathbf{a}}_\rho)^{d_\rho+k}$ contain a closed strongly positive form and belong to $\Pi(E)$.*

We obtain thus a partial affirmative answer to Griffiths' question for the polynomials in the Chern forms of (E,h) belonging to the positive convex sub-cone $\mathcal{F}(E,h) \subset \Pi(E,h)$ spanned by *all possible wedge products of all possible push-forwards* $(\pi_\rho)_*c_1(Q^{\mathbf{a}}_\rho, h)^{\wedge(d_\rho+k)}$, for $k = 0,\dots,n$, as the weights $\mathbf{a} \in \mathbb{N}^r$ vary in the appropriate range prescribed by Conditions (1.9) and (1.12).

This sub-cone contains in particular the signed Segre forms, which arise in the case of projectivized bundle.

Remark 3.9. Even if it is possible to obtain every Schur form as a push-forward, see Remark 2.10 and Proposition 2.19, this is not enough to get here that every Schur form of a Griffiths semipositive vector bundle is positive. This is because the curvature computations for the tautological bundles over the compete flag bundle do not permit to conclude that the relevant monomials whose push-forward give the desired Schur forms are positive.

The next examples are intended to give some flavor of which kind of new positive forms, in particular besides signed Segre forms, we are able to obtain with our methods.

3.2.1 Examples

In this section we give several examples of differential forms whose (strong) positivity is due to Theorem 3.6 and was not previously known in general. The explicit forms of some of them are obtained by implementing Formula (2.1) in PARI/GP as illustrated in Appendix A.

As we have seen, for (E, h) a rank r Griffiths (semi)positive vector bundle, the forms $(\pi_\rho)_* c_1(Q_\rho^{\mathbf{a}}, h)^{\wedge(d_\rho+k)}$ are strongly positive. We want to highlight here some among them that cannot be shown to be positive only using results in the literature preceding the present work, at the best of our knowledge.

To this aim, observe that we already knew that the $(-1)^k s_k(E, h)$'s, *i.e.*, the signed Segre forms, are (strongly) positive for Griffiths (semi)positive vector bundles thanks to [Gul12, Theorem 1.1] (even though the strong positivity was not explicitly observed there). Also, as already noted, the product of positive forms, all of them strongly positive (resp. all except possibly one) is strongly positive (resp. positive).

This understood, in order to check for which forms we get new information about their positivity, we shall express $(\pi_\rho)_* c_1(Q_\rho^{\mathbf{a}}, h)^{\wedge(d_\rho+k)}$ as a polynomial in the (signed) Segre forms of (E, h).

In what follows, in order to simplify the notation, we denote by $c_1, \dots, c_r$ the Chern forms of (E, h) and by $s_1, \dots, s_n$ the Segre forms of (E, h). The symbol S_σ stands for the Schur form of (E, h) associated to the partition σ. Moreover, we omit the symbol $\wedge$ for the wedge product of forms.

Push-forwards from Grassmannian bundles

Denote by ρ the sequence $(0, r-d, r)$. Then, $\mathbb{F}_\rho(E)$ is the Grassmannian bundle $\mathbb{G}_{r-d}(E)$ of $(r-d)$-planes in E. Let $\pi \colon \mathbb{G}_{r-d}(E) \to X$ be the projection, and denote by Q the universal quotient bundle of rank d on $\mathbb{G}_{r-d}(E)$ equipped with the quotient metric. In our notation, the class $c_1(Q)$ equals $c_1(Q_{\rho,1})$.

Therefore, for $N \geq d(r-d)$ the metric counterpart of the Darondeau–Pragacz push-forward formula reads

$$\pi_* c_1(Q, h)^N = \sum_{|\lambda|=N-d(r-d)} f^{\lambda+\varepsilon} \det\Big((-1)^{\lambda_i+j-i} s_{\lambda_i+j-i}\Big)_{1\leq i,j\leq d} \tag{3.1}$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$ is a partition and $|\lambda|$ is its total weight, ε stands for the d-uple $(r-d)^d = (r-d, \dots, r-d)$ and $f^{\lambda+\varepsilon}$ is the number of standard Young tableaux with shape $\lambda+\varepsilon$ (we have used here the more explicit version computed in the particular case of Grassmannian bundles in [KT15, Theorem 0.1]). Now, as explained for instance in [KT15], we have that

$$f^{\lambda+\varepsilon} = \frac{N! \prod_{1\le i<j\le d}(\lambda_i - \lambda_j - i + j)}{\prod_{1\le i\le d}(r + \lambda_i - i)!}.$$

Note that when $d = 1$ the bundle $\mathbb{G}_{r-1}(E)$ can be identified with $\mathbb{P}(E^\vee)$, consequently $Q \cong \mathcal{O}_{\mathbb{P}(E^\vee)}(1)$, and Formula (3.1) becomes

$$\pi_* c_1(\mathcal{O}_{\mathbb{P}(E^\vee)}(1), h)^N = (-1)^{N-r+1} s_{N-r+1},$$

which is the push-forward formula by [Mou04, Gul12, Div16] giving the positivity of signed Segre forms.

It is noteworthy (see Example 1.8) to observe that already for the projectivized bundle of lines $\mathbb{P}(E)$ corresponding to the partition $(0, 1, r)$, if we push-forward powers of $c_1(Q)$, where $Q = \pi^* E/\mathcal{O}_{\mathbb{P}(E)}(-1)$, we are now able to get forms whose positivity was not previously known.

In rank 3 (if $r = 2$ we have that $\mathbb{P}(E) \cong \mathbb{P}(E^\vee)$ and there is nothing more to add) we see for instance, by using Formula (3.1), that

$$\pi_* c_1(Q, h)^5 = 4c_1^3 - 3c_1c_2 - c_3 = s_3 - 5s_1s_2,$$

and the positivity of $s_3 - 5s_1s_2$ was not previously known given that s_3 is negative.

Analogously, for the same reasons, if $r = 4$, the positivity of

$$\pi_* c_1(Q, h)^6 = 10c_1^3 - 4c_1c_2 - c_3 = s_3 - 6s_1s_2 - 5s_1^3$$

was not previously known.

The simplest example of a Grassmannian bundle which is not a projectivized bundle is $\mathbb{G}_2(E)$ for E of rank 4. Also in this case we get something new, as follows. By Formula (3.1), the push-forward *via* π of $c_1(Q, h)^N$ is given by

$$\begin{cases} 2 & \text{for } N = 4, \\ 5c_1 & \text{for } N = 5, \\ 9c_1^2 - 4c_2 & \text{for } N = 6, \\ 14(c_1^3 - c_1c_2) & \text{for } N = 7, \\ 2(10c_1^4 - 16c_1^2c_2 - c_1c_3 + 3c_2^2 + 4c_4) & \text{for } N = 8. \end{cases}$$

When rewritten in terms of signed Segre forms, we obtain

$$\begin{cases} 2 & \text{for } N = 4, \\ 5(-s_1) & \text{for } N = 5, \\ 5s_1^2 + 4s_2 & \text{for } N = 6, \\ 14(-s_1)s_2 & \text{for } N = 7, \\ 2(7s_1s_3 + 7s_2^2 - 4s_4) & \text{for } N = 8, \end{cases}$$

so that the positivity of the last form could not be previously deduced, since $-s_4$ is negative.

Push-forwards from complete flag bundles: the case of rank 3

The general formulæ for the push-forwards of $c_1(Q^{(a,b,c)},h)^{3+k}$ in terms of Schur forms, up to degree 3, are:

$$
\begin{aligned}
k=0 &\quad\rightsquigarrow\quad 3(a^2b-ab^2-a^2c+ac^2+b^2c-bc^2),\\
k=1 &\quad\rightsquigarrow\quad 4(a^3b-ab^3-a^3c+ac^3+cb^3-bc^3)S_{(1)},\\
k=2 &\quad\rightsquigarrow\quad 10(a^3b^2-a^2b^3-a^3c^2+a^2c^3+b^3c^2-b^2c^3)S_{(2,0)}\\
&\qquad\quad +5(a^4b-ab^4-a^4c+ac^4+b^4c-bc^4)S_{(1,1)},\\
k=3 &\quad\rightsquigarrow\quad 60(a^3b^2c-a^2b^3c-a^3bc^2+a^2bc^3+ab^3c^2-ab^2c^3)S_{(3,0,0)}\\
&\qquad\quad +15(a^4b^2-a^2b^4-a^4c^2+a^2c^4+b^4c^2-b^2c^4)S_{(2,1,0)}\\
&\qquad\quad +6(a^5b-ab^5-a^5c+ac^5+b^5c-bc^5)S_{(1,1,1)}.
\end{aligned}
$$

Clearly, our method produces positive forms until k reaches n, but already from these first cases we see how the complexity rapidly increases. Thanks to Theorem 3.6 we can say that all of the above listed forms belong to the positive cone for every $a \geq b \geq c \geq 0$.

For a concrete example, if $(a,b,c)=(3,2,0)$ we obtain in terms of Segre forms:

$$
\begin{aligned}
&\pi_* c_1(Q^{(3,2,0)},h)^6\\
&= 2700\, S_{(2,1,0)} + 2340\, S_{(1,1,1)}\\
&= 180(-15 s_1 s_2 + 2 s_3),
\end{aligned}
$$

and the positivity of this form was not already known since s_3 is negative.

Note that, for instance, if $(a,b,c)=(2,1,0)$ the positivity of

$$
\begin{aligned}
&\pi_* c_1(Q^{(2,1,0)},h)^6\\
&= 180\, S_{(2,1,0)} + 180\, S_{(1,1,1)}\\
&= -180 s_1 s_2
\end{aligned}
$$

was instead previously known since $-s_1 s_2$ is the wedge product of strongly positive forms.

Push-forwards from complete flag bundles: the case of rank 4

Here, we prefer to emphasize the different behavior in two special cases instead of giving the general formulæ for $a \geq b \geq c \geq d \geq 0$.

For $(a,b,c,d)=(3,2,1,0)$, we obtain

$$
\begin{aligned}
\pi_* c_1(Q^{(3,2,1,0)},h)^9 &= 90720(-s_1^3 - 2 s_1 s_2),\\
\pi_* c_1(Q^{(3,2,1,0)},h)^{10} &= 5040(216 s_1^2 s_2 + 7 s_1 s_3 + 39 s_2^2 - 4 s_4).
\end{aligned}
$$

Note that the positivity of the last form was not previously known since $-s_4$ is negative, while we already knew that $-s_1^3 - 2s_1 s_2$ is positive.

Now set $(a,b,c,d)=(4,3,2,0)$. In this case, we get

$$
\begin{aligned}
\pi_* c_1(Q^{(4,3,2,0)},h)^9 &= 181440(-8 s_1^3 - 12 s_1 s_2 + s_3),\\
\pi_* c_1(Q^{(4,3,2,0)},h)^{10} &= 40320(648 s_1^2 s_2 - 124 s_1 s_3 + 42 s_2^2 + 13 s_4).
\end{aligned}
$$

The positivity of both of these forms was not already known again because s_3 is negative.

3.3 Other positive characteristic forms

In this section we present a variant of the approach in Section 1.4.1, in order to obtain positivity of some Schur forms which, in general, are not in the cone $\mathcal{F}(E,h)$.

More specifically, we show the positivity of the Schur form c_2 in every rank and $c_1c_2 - c_3$ in rank 3 for Griffiths (semi)positive Hermitian holomorphic vector bundles.

In what follows, we proceed to observe that the positivity of the second Chern form is implied by the Griffiths *semi*positivity of the curvature tensor. Such argument is central in the proof of the positivity of $c_1c_2 - c_3$.

Let (E,h) be a Griffiths positive Hermitian holomorphic vector bundle of rank 2 over a complex manifold X. Thanks to [Gri69, Appendix to §5.(b)], we know that the second Chern form $c_2(E,h)$ is positive. The argument used by Griffiths in the proof consists in the following. First, by Proposition 1.23, one can assume that X is a complex surface. Of course, this assumption is not allowed if one wants to show the Hermitian (or strong) positivity of $c_2(E,h)$ (cf. with Remarks 1.19 and 1.22). However, in rank 2 the curvature is a 2×2 matrix of $(1,1)$-forms, thus

$$c_2(E,h) = -\frac{1}{4\pi^2} \det \Theta(E,h).$$

Given that the Hermitian forms associated to the diagonal entries of the matrix $i\Theta(E,h)$ are positive definite, one can perform a simultaneous diagonalization of such Hermitian forms. After the diagonalization, using that $\dim X = 2$, to show that $-\det \Theta(E,h)$ is positive it is sufficient to apply the Schwarz inequality coupled with the definition of Griffiths positivity.

Now, suppose that (E,h) is Griffiths semipositive. Once fixed a strictly positive $(1,1)$-form ω on X and $\varepsilon > 0$, we can apply the above argument to the $\mathrm{Herm}(E,h)$-valued $(1,1)$-form

$$i\Theta(E,h) + \varepsilon\omega \otimes \mathrm{Id}_E \,.$$

Passing to the limit for $\varepsilon \to 0$ we get the following.

Lemma 3.10. *Given $(E,h) \to X$ Griffiths semipositive of rank 2, the differential form $c_2(E,h)$ is positive.*

3.3.1 Positivity of c_2 in any rank

Let $(E,h) \to X$ be a Hermitian holomorphic vector bundle of rank r, and denote by $\Theta_{\alpha\beta}$ the entries of the curvature matrix. In general, we have that

$$\begin{aligned} c_2(E,h) &= \mathrm{tr}_{\mathrm{End}(\Lambda^2 E)} \left(\bigwedge^2 \frac{i}{2\pi} \Theta(E,h) \right) \\ &= -\frac{1}{4\pi^2} \sum_{1 \le \alpha < \beta \le r} (\Theta_{\alpha\alpha} \wedge \Theta_{\beta\beta} - \Theta_{\alpha\beta} \wedge \Theta_{\beta\alpha}). \end{aligned} \tag{3.2}$$

If (E, h) is Griffiths positive, every diagonal entry $\Theta_{\alpha\alpha}$ of $\Theta(E, h)$ gives a positive definite Hermitian form. Therefore, it is possible to apply [Gri69, Appendix to §5.(b)] to all the summands in the last member of Formula (3.2). This because each summand is by definition the determinant of a 2×2 principal sub-matrix of the curvature matrix. From this follows the positivity of $c_2(E, h)$ in any rank.

In the semipositive case, the positivity of $c_2(E, h)$ in any rank is obtained by applying Lemma 3.10 (instead of the above argument) to all the summands in the last member of Formula (3.2). This proves the following.

Theorem 3.11. *Let (E, h) be a Griffiths semipositive Hermitian holomorphic vector bundle over a complex manifold. Then the second Chern form*

$$S_{(2,0)}(E, h) = c_2(E, h)$$

is a positive $(2, 2)$-form.

3.3.2 Positivity of $c_1c_2 - c_3$ in rank 3

From now on, suppose that $r = 3$. Let $p := \pi_{(0,1,3)} \colon \mathbb{P}(E) \to X$ be the projective bundle of lines in E with associated tautological short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow p^*E \twoheadrightarrow Q := p^*E/\mathcal{O}_{\mathbb{P}(E)}(-1) \to 0$$

over $\mathbb{P}(E)$. Note that the quotient bundle $Q \to \mathbb{P}(E)$ is Griffiths semipositive of rank 2 with respect to the natural quotient metric induced by (E, h) (and denoted again by h). Consequently, the $(1, 1)$-form $c_1(Q, h)$ is strongly positive and, by Lemma 3.10, $c_2(Q, h)$ is positive.

We are now ready to prove the following result.

Theorem 3.12. *Let (E, h) be a Griffiths semipositive Hermitian holomorphic vector bundle of rank 3 over a complex manifold. Then the Schur form*

$$S_{(2,1,0)}(E, h) = c_1(E, h) \wedge c_2(E, h) - c_3(E, h)$$

is a positive $(3, 3)$-form.

Remark 3.13. The main idea behind the next proof was developed for the first time by the author in an alternative proof of the positivity of $c_2(E, h)$ in rank 3. Since the positivity of $c_2(E, h)$ for Griffiths semipositive vector bundles is the content of Theorem 3.11, we omit such an alternative proof.

Proof of Theorem 3.12. Let $\pi \colon \mathbb{F}(E) \to X$ be the complete flag bundle associated to $E \to X$.

Following the notation of Section 2.4, let ξ_1, ξ_2, ξ_3 be the Chern roots of $\pi^*E^\vee$. They are given by the tautological filtration (1.1) (where $r = m = 3$), for which π^*E splits (non canonically) as a differentiable vector bundle over $\mathbb{F}(E)$ as $Q_1 \oplus Q_2 \oplus Q_3$.

By the commutativity of the following diagram of projections

$$\begin{array}{ccc} \mathbb{F}(E) & \xrightarrow{q := \pi^{(0,1,2,3)}_{(0,1,3)}} & \mathbb{P}(E) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array} \tag{3.3}$$

(cf. with the commutative diagram (1.5) above) we see that ξ_1 and ξ_2 are also the Chern roots of $q^*Q^\vee$, where $Q \to \mathbb{P}(E)$ is the rank 2 tautological quotient bundle introduced before. Indeed,

$$q^*Q = q^*\left(p^*E/\mathcal{O}_{\mathbb{P}(E)}(-1)\right) = \pi^*E/U_1$$

and, since $\mathbb{F}(E)$ coincides with $\mathbb{F}(Q) = \mathbb{P}(Q)$, we deduce that there is a non canonical isomorphism

$$q^*Q \cong_{C^\infty} \pi^*E/U_2 \oplus U_2/U_1 = Q_1 \oplus Q_2.$$

By Formula (2.19) applied to $\pi\colon \mathbb{F}(E) \to X$ and where $F(\Xi_1, \Xi_2, \Xi_3)$ is taken to be the monomial $\Xi_1^{\wedge 4} \wedge \Xi_2^{\wedge 2} \wedge \Xi_3^{\wedge 0}$, we have the equality

$$\pi_*[\Xi_1^{\wedge 4} \wedge \Xi_2^{\wedge 2} \wedge \Xi_3^{\wedge 0}] = s_{(-2,1,4)}(E,h) = S_{(2,1,0)}(E,h). \tag{3.4}$$

Observe however that, *a priori*, we do not conclude anything about the positivity of $S_{(2,1,0)}(E,h)$ using only Equation (3.4), since, for instance, the line bundle Q_2 is not positive in general. Therefore, we apply again Formula (2.19), but to the flag bundle $q\colon \mathbb{P}(Q) \to \mathbb{P}(E)$, getting

$$q_*[\Xi_1^{\wedge 4} \wedge \Xi_2^{\wedge 2}] = s_{(1,4)}(Q,h) = c_1(Q,h) \wedge c_2(Q,h)^{\wedge 2}, \tag{3.5}$$

where the last equality holds given that $c_3(Q,h) \equiv 0$ and $c_4(Q,h) \equiv 0$.

Hence, the chain of equalities

$$\begin{aligned} p_*\left[c_1(Q,h) \wedge c_2(Q,h)^{\wedge 2}\right] &= p_* q_*[\Xi_1^{\wedge 4} \wedge \Xi_2^{\wedge 2}] \\ &= \pi_*[\Xi_1^{\wedge 4} \wedge \Xi_2^{\wedge 2} \wedge \Xi_3^{\wedge 0}] \quad \text{by commutativity of (3.3)} \\ &= S_{(2,1,0)}(E,h) \quad \text{by Formula (3.4)} \end{aligned}$$

follows, by applying p_* to both members of Formula (3.5).

Since $Q \to \mathbb{P}(E)$ is Griffiths semipositive, by Lemma 3.10 we know that $c_2(Q,h)$ is a positive form on $\mathbb{P}(E)$. Thanks to [BP13, Theorem 1], we have that the square of a positive $(2,2)$-form is also positive. Hence $c_1(Q,h) \wedge c_2(Q,h)^{\wedge 2}$ is positive, given that it is the wedge product of a strongly positive and of a positive form. Consequently, $S_{(2,1,0)}(E,h)$ must be a positive form on X, since it is the push-forward of a positive differential form. □

Remark 3.14. For dimension and bi-degree reasons (see Remark 1.19) we underline that if $\dim X \leq 3$, then $c_2(E,h)$ is strongly positive, and if $\dim X \leq 4$, then $S_{(2,1,0)}(E,h)$ is strongly positive, too.

Remark 3.15. It is interesting to observe that Theorem 3.12 can be deduced also by Theorem 2.16. More precisely, the general push-forward formula obtained in Theorem 2.16 gives us the equality

$$p_*\left[c_1(Q,h) \wedge c_2(Q,h)^{\wedge 2}\right] = S_{(2,1,0)}(E,h)$$

without the need to use the map π_* of the complete flag bundle. However, the previous strategy of proof has provided us the right (positive) differential form on $\mathbb{P}(E)$ to be pushed-forward to obtain the positivity of $S_{(2,1,0)}(E,h)$.

From the (strong) positivity of the second Segre form

$$s_2(E,h) = c_1(E,h)^{\wedge 2} - c_2(E,h)$$

(see [Gul12, Theorem 1.1] and Theorem 3.6), and from Theorem 3.12 we deduce the following.

Corollary 3.16. *If (E,h) is Griffiths semipositive of rank 3 over a complex manifold, then the following pointwise inequalities hold*

$$c_1(E,h)^{\wedge 3} \geq c_1(E,h) \wedge c_2(E,h) \geq c_3(E,h). \tag{3.6}$$

In addition, if X is compact and 3-dimensional, the following chain of inequalities between Chern numbers of E also holds

$$\int_X c_1(E)^3 \geq \int_X c_1(E)c_2(E) \geq \int_X c_3(E). \tag{3.7}$$

As already mentioned in the introduction, the inequalities in (3.6) can be deduced by [Li21, Theorem 3.2] only if h is dual Nakano semipositive. Thus, Corollary 3.16 is in some sense a generalization of [Li21, Theorem 3.2] in rank 3.

Remark 3.17. It is well-known that a vector bundle admitting a Griffiths semipositive metric is nef (see [DPS94] for a definition). Recall that the Chern numbers of a nef vector bundle on a compact n-dimensional Kähler manifold are bounded above by the Chern number c_1^n ([DPS94, Corollary 2.6]) and below by the Euler number c_n ([LZ20, Theorem 2.9]). See also [Li21, Remark 3.3]. Therefore, assuming that X is also Kähler the first and the second inequalities in (3.7) are a particular case of [DPS94, Corollary 2.6] and [LZ20, Theorem 2.9] respectively. Moreover, observe that the inequalities in (3.7) follow from [DPS94, Theorem 2.5] if X is Kähler.

Remark 3.18. It seems difficult to apply the strategy in the proof of Theorem 3.12 to prove the positivity of $c_3(E,h)$ in rank 3. Indeed, in order to factor through the quotient bundle, we need to push-forward only monomials of the form $\Xi_1^{\wedge\lambda_1} \wedge \Xi_2^{\wedge\lambda_2}$. But, by Formula (2.19) we have the equality $\pi_*[\Xi_1^{\wedge 3} \wedge \Xi_2^{\wedge 2} \wedge \Xi_3] = c_3(E,h)$ that involves Ξ_3, which is the pull-back through q of the Chern curvature of $\mathcal{O}_{\mathbb{P}(E)}(-1)$.

However, one can try to adapt the same ideas presented in Theorem 3.12 to vector bundles of rank higher than 3. For example, in rank 4 we could use the tautological quotient Q of the Grassmann bundle $\mathbb{G}_2(E)$ in order to get push-forward formulæ for some Schur forms. The problem here is that the monomials we push-forward contain $c_2(Q,h)^{\wedge\ell}$, $\ell > 2$. Therefore, as pointed out for instance in [BP13], we do not know *a priori* if these powers are positive.

3.4 State of the art, remarks and open questions

In this concluding section we suppose, as always, that (E,h) is a rank r Hermitian holomorphic vector bundle over a complex manifold X.

3.4.1 A comparison on the positivity of Schur forms

We compare here the different notions of positivity (see Definition 1.15) of Schur forms that appear in [BC65, Gri69, Gul12, Li21, Fin21], which are the works mainly related to Theorem C and Theorem D.

Recall that we do not deal with the strict notions of positivity (see Remark 1.17). For this, the following exposition can be made slightly more precise, although this does not affect the purpose of the comparison.

Positivity in [BC65]

In [BC65, Definition 5.1] a notion of positivity for elements in $\mathcal{A}^{p,p}(X, \mathrm{End}(E))$ is given. In particular, for $p = 1$, such notion includes the so-called Bott–Chern nonnegativity (this terminology is due to [Li21], see below), which requires that the curvature of (E, h) can be expressed, locally, as $A \wedge \bar{A}^t$, where A is a matrix of $(1, 0)$-forms of appropriate size. Although [BC65] predates Griffiths' conjecture, from [BC65, Lemma 5.3] one can deduce the Hermitian positivity of the top Chern form of a Bott–Chern nonnegative vector bundle.

Positivity in [Gri69]

The positivity notion considered in [Gri69, p. 240] is, by Proposition 1.21, Hermitian positivity. As already mentioned, Griffiths' conjecture first appears in [Gri69, p. 247], where it is conjectured that for a Griffiths positive vector bundle the cone of positive polynomials in the Chern forms (*i.e.*, the Schur cone) consists of Hermitian positive differential forms. The full conjecture is verified in [Gri69, p. 246] for globally generated vector bundles; for more details we refer to [Gri69, Proof of Theorem D].

Moreover, by the characterization of Proposition 1.23, Griffiths proves in [Gri69, Appendix to §5.(b)] the positivity, but not the Hermitian positivity, of $c_2(E, h)$, for (E, h) Griffiths positive of rank 2.

Positivity in [Gul12]

The main result of [Gul12] states that the signed Segre forms of a Griffiths positive vector bundle are positive. Actually, [Gul12, Theorem 1.1] implicitly proves that such forms are strongly positive, even though the strong positivity is not explicitly observed therein.

Positivity in [Li21]

The two positivity notions considered by [Li21] are, by Propositions 1.23 and 1.21, positivity and Hermitian positivity, although in [Li21, §2] they are called *nonnegativity* and *strong nonnegativity* respectively. In the same spirit of [Gri69, Proof of Theorem D], [Li21, Proposition 3.1] extends the above mentioned result on globally generated vector bundles to the larger family of Bott–Chern nonnegative (see [Li21, Definition 2.1]) vector bundles, showing that the Schur forms of these bundles are Hermitian positive.

Positivity in [Fin21]

All the three notions of Definition 1.15 are addressed in [Fin21], although with the terminology of [HK74]. By assuming Nakano, or dual Nakano, (semi)positivity [Fin21, Theorem 1.1] shows that Schur forms are Hermitian positive.

It is also worth to recall that [Fin21] relates, in a very interesting way, Bott–Chern notion of positivity with the dual Nakano one. More precisely, [Fin21, Theorem 2.14] states that a Hermitian vector bundle $(E, h) \to X$ is dual Nakano semipositive if and only if for every $x \in X$ there is a vector space V and a matrix of $(1, 0)$-forms $A \in T^{\vee}_{X,x} \otimes \mathrm{Hom}(V, E_x)$ such that $\Theta(E, h)_x = A \wedge \bar{A}^t$. In order to have a similar characterization for dual Nakano positivity, we observe that it is sufficient to require, in addition, the invertibility of the operator $\tilde{A} \in \mathrm{Hom}(T_{X,x} \otimes E^{\vee}_x, V^{\vee})$ naturally associated to A. Therefore, both [Li21] and [Fin21] obtain the full Griffiths' conjecture for dual Nakano (semi)positive bundles, but with different methods.

Finally, we recall that [Fin21, Theorem 1.3] establishes the equivalence between Griffiths' conjecture and an open question concerning *mixed discriminants*, see [Fin21, Open problem].

Positivity in this thesis

Given a Griffiths semipositive vector bundle $(E, h) \to X$, Theorem 3.6 shows the strong positivity of the family $\mathcal{F}(E, h)$ of differential forms in the Schur cone $\Pi(E, h)$. Such family form a sub-cone which, in the notation of Section 3.2, is spanned by all possible wedge products of all possible push-forwards

$$(\pi_\rho)_*(a_1\Xi_{\rho,1} + \cdots + a_m\Xi_{\rho,m})^{\wedge(d_\rho+k)} \tag{3.8}$$

as $k \geq 0$, $\rho = (\rho_0, \ldots, \rho_m)$ and the sequence of integers $a_1 \geq \cdots \geq a_m \geq 0$ vary. As already mentioned, observe that to $\rho = (0, r-1, r)$ corresponds the bundle $\pi_\rho \colon \mathbb{P}(E^{\vee}) \to X$ of hyperplanes in E. Hence $\Xi_{\rho,1} = c_1(\mathcal{O}_{\mathbb{P}(E^{\vee})}(1), h)$ and for $(a_1, a_2) = (1, 0)$ Guler's result [Gul12, Theorem 1.1] on the signed Segre forms is recovered. Moreover, Theorems 3.11 and 3.12 show the positivity of $c_2(E, h)$ in any rank and $S_{(2,1,0)}(E, h)$ in rank 3 respectively.

Currently, the positivity of other (positive combinations of) Schur forms of Griffiths semipositive bundles is not known.

3.4.2 Concluding remarks

If we assume (dual) Nakano semipositivity, then the Schur cone consists of Hermitian positive differential forms. Hence, it is natural to ask the following question.

Question 2. Beside those as in Expression (3.8) (and wedge products of them), are there in the Schur cone other strongly positive differential forms for (dual) Nakano semipositive vector bundles?

Of course, the best possible result would be that all the Schur cone consists of strongly positive differential forms.

Assume now that the vector bundle is Griffiths semipositive. Although it is not explicitly mentioned in Griffiths' conjecture (as stated in the literature), by the results listed in Section 3.4.1 it seems interesting to ask what is the most natural notion of positivity, among those in Definition 1.15, that we can expect to hold for the Schur forms of the vector bundle. Certainly, the conjecture is still open even requiring the weakest notion of positivity, although, as already said, Theorem 3.6 shows us that the strong positivity naturally appears in this context. In addition, it is in a sense more natural to require the strong, or Hermitian, positivity of the Schur forms (see Remark 3.7). Indeed, as mentioned in Section 1.3, positive polynomials (*i.e.*, those belonging to the Schur cone) are stable under product and so do Hermitian and strongly positive forms; while the wedge product of two positive forms is not necessarily positive (see [Dem12, §III, (1.11) Proposition] and [BP13]).

Summing up, one may ask an analogue of Question 2 for Griffiths semipositive vector bundles, wondering if, outside the cone $\mathcal{F}(E,h)$ spanned by the wedge products of the push-forwards (3.8), there are other Hermitian or strongly positive Schur forms. If they were all (at least) Hermitian positive one would have an affirmative answer to the original Griffiths' conjecture as stated in [Gri69, p. 247].

However, this question is currently open. For instance, beside the cases listed in Remark 3.14, we do not deduce the Hermitian positivity of $c_2(E,h)$ (resp. of $S_{(2,1,0)}(E,h)$) by the proof of Theorem 3.11 (resp. of Theorem 3.12).

3.4.3 Further related questions

Let $\mathcal{V}$ be an ample vector bundle over a projective manifold X.

As pointed out in [Pin18, Xia22], one can study some variants of Griffiths' conjecture. For instance, [Xia22, Conjecture 1.4] asks whether every Schur class $S_\sigma(\mathcal{V})$ does admit a positive representative. Such question fits between Fulton–Lazarsfeld theorem and Griffiths' conjecture. Thanks to [Xia22, Theorem A] we know that the answer to this question is affirmative if $|\sigma| = \dim X - 1$. Another partial affirmative answer to [Xia22, Conjecture 1.4] is given by our Theorem 3.8, where it is shown that the positive combinations of Schur classes given by push-forwards as in Expression (3.8) contain a strongly positive form. However, the general problem is open.

Moreover, as observed in [Xia22, Remark 1.5], given a smooth representative $\eta \in S_\sigma(\mathcal{V})$ it is not clear how to find a Hermitian metric h on $\mathcal{V}$ such that $\eta = S_\sigma(\mathcal{V},h)$. By virtue of these remarks, one may wonder the following.

Question 3. Given an ample vector bundle $\mathcal{V} \to X$, is it true that for any partition $\sigma \in \Lambda(k,r)$, there exists a Hermitian metric h_σ on $\mathcal{V}$ such that $S_\sigma(\mathcal{V},h_\sigma)$ is (Hermitian/strongly) positive?

If $\dim X = 2$ and $\mathcal{V}$ is semistable with respect to some polarization, [Pin18, Theorem 1.1] provides an affirmative answer to Question 3, finding a Hermitian metric on $\mathcal{V}$ whose Schur forms are positive. However, it is not clear if such metric is Griffiths positive: see [Pin18, p. 633], and also [Fin21, Section 4.2] for related considerations. Clearly, an affirmative answer to Question 3 would imply [Xia22, Conjecture 1.4].

We would finally remark that affirmative answers to [Xia22, Conjecture 1.4] or Question 3 do not imply Griffiths' conjecture, given that the latter requires the positivity of the Schur forms with respect to one *given* Griffiths (semi)positive Hermitian metric on the vector bundle.

Appendix A

PARI/GP computations

Here we report the codes implemented on PARI/GP CALCULATOR Version 2.11.4 which provide some of the explicit examples of strongly positive differential forms given in Section 3.2.1. These codes are implementations of the right hand side of Formula 2.1 in case of complete flag bundles.

Although appropriate simple modifications of the following codes provide all the differential forms obtainable from Theorem C in case of complete flag bundles, the computation complexity rapidly increases. For instance, our computers reached the general formulæ in the case of rank 4 in a long time.

Note that all the outputs of the following codes are in terms of Segre forms $s_1, \dots, s_n$. To obtain outputs in terms of Schur forms we have implemented change of basis formulæ up to degree 4, which we give at the end of this section.

Push-forwards from complete flag bundles: $r = 3$, $n = 3$**:**

```
t=[t1,t2,t3]; /*formal variables*/
s=[s1,s2,s3]; /*Segre forms of (E,h) on X*/
q=prod(i=1,3, 1+sum(k=1,3, (1/((t[i])^k))*s[k]));
v=(t[1]-t[2])*(t[1]-t[3])*(t[2]-t[3]); /*Vandermonde polynomial*/
Y=q*v;
w=[a,b,c]; /*exponents of the universal line bundle*/

/*the following must be implemented on the same line*/
for(d=3,6,
C=((w[1]*(-t[1])+w[2]*(-t[2])+w[3]*(-t[3]))^d)*Y;
for(j=1,3, C=polcoeff(C,2,t[j]));
print(C))
```

Push-forwards from complete flag bundles: $r = 4$, $n = 4$**:**

```
t=[t1,t2,t3,t4]; /*formal variables*/
s=[s1,s2,s3,s4]; /*Segre forms of (E,h) on X*/
q=prod(i=1,4, 1+sum(k=1,4, (1/((t[i])^k))*s[k]));
v=prod(i=1,3, prod(j=i+1,4, t[i]-t[j])); /*Vandermonde polynomial*/
Y=q*v;
```

```
w=[a,b,c,d]; /*exponents of the universal line bundle*/

/*the following must be implemented on the same line*/
for(d=6,10,
C=((w[1]*(-t[1])+w[2]*(-t[2])+w[3]*(-t[3])+w[4]*(-t[4]))^d)*Y;
for(j=1,4, C=polcoeff(C,3,t[j]));
print(C))
```

Remark A.1. Of course, the implementations above work for any chosen dimension n simply by modifying the array `s` of the Segre forms and the polynomial `q` accordingly. Consequently, by modifying the upper limit of `d` (which is the relative dimension of the complete flag bundle) in the last line of the code we get $n+1$ outputs.

Change of basis from Segre forms to Schur forms

Given C a weighted homogeneous polynomial of degree d in the Segre forms of (E, h), we want an array S containing all the coefficients of C in terms of the basis given by Schur forms. In order to do this, the code we have implemented is the following.

```
/*d=1*/
S=[-polcoeff(C,1,s1)]

/*d=2*/
S=[polcoeff(C,2,s1),polcoeff(C,2,s1)+polcoeff(C,1,s2)]

/*d=3*/
/*the following must be implemented on the same line*/
S=
[
-polcoeff(C,3,s1),
-2*polcoeff(C,3,s1)-polcoeff(polcoeff(C,1,s1),1,s2),
-polcoeff(C,3,s1)-polcoeff(polcoeff(C,1,s1),1,s2)-polcoeff(C,1,s3)
]

/*d=4*/
/*the following must be implemented on the same line*/
S=
[
polcoeff(C,4,s1),
3*polcoeff(C,4,s1)+polcoeff(polcoeff(C,2,s1),1,s2),
2*polcoeff(C,4,s1)+polcoeff(polcoeff(C,2,s1),1,s2)+polcoeff(C,2,s2),
3*polcoeff(C,4,s1)+2*polcoeff(polcoeff(C,2,s1),1,s2)+
polcoeff(polcoeff(C,1,s1),1,s3)+polcoeff(C,2,s2),
polcoeff(C,4,s1)+polcoeff(polcoeff(C,2,s1),1,s2)+
polcoeff(polcoeff(C,1,s1),1,s3)+polcoeff(C,2,s2)+polcoeff(C,1,s4)
]
```

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