HODGE THEORY AND TAME GEOMETRY (ABSTRACT FOR A 20 HOURS SERIES OF LECTURES)

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Complex Hodge theory is a powerfull tool for analyzing the geometry of complex algebraic varieties and of their moduli spaces. However, the parameter spaces for Hodge structures are complex analytic spaces that are only rarely algebraic, and this is the source of many complications in the theory. Nevertheless, it has been recently realized that the constructions of period spaces and period maps in variational Hodge theory take naturally place in the intermediate category of so-called $\mathbb{R}_{an,exp}$-definable complex analytic spaces, which enjoy both some of the local flexibility of analytic varieties and some of the global rigidity of algebraic varieties. A good illustration of this behaviour is provided by the following o-minimal Chow theorem of Peterzil and Starchenko [PS09]: A closed $\mathbb{R}_{an,exp}$-definable complex analytic subspace of a complex algebraic variety is automatically algebraic, whether the ambient space is proper or not.

The goal of these series of lectures is to give a survey of recent advances in Hodge theory via o-minimal techniques aimed at non-experts. I will first give an introduction to o-minimal geometry from a geometric point of view, and then explain the basics of definable complex analytic geometry with a view towards algebraization theorems. Among the Hodge-theoretic applications, I will prove the algebraicity of Hodge loci and the quasiprojectivity of images of period maps, following [BKT20, BBT19, BBKT20, BBT20]. Here is a more precise content of the lectures:

Tame analytic geometry and algebraization.

- The general tameness properties of o-minimal structures
- o-minimality of $\mathbb{R}_{an}$ and $\mathbb{R}_{an,exp}$
- Definable complex analytic spaces and definable coherent sheaves
- o-minimal Chow and o-minimal GAGA

Applications to Hodge theory.

- Variations of Hodge structures and period maps
- Definability of uniformizations of Shimura varieties and period maps
- Applications to the algebraicity of Hodge loci and of images of period maps.

References

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